# Chapter 1 Introduction

- 1.1 Background
  In trying to understand physical phenomena, models yield equations that have derivatives or unknown functions called differential equations
- $t$ is the dependent variable $x$ is the independent variable, $a$, $k$ are coefficients.
- Ordinary differential equation one with a single independent variable
- Partial Differential equation one with multiple independent variables
- Order of a differential equation is the order of the highest derivative.

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0 \quad \text{: order } 2^{nd}$$

- Linear differential equation: one that has one dependent variable $y$ and it's derivatives appear in additive combinations of first powers

$$a_n x \frac{d^n y}{dx^n} + a_{n-1} x \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_0 x y = F(x)$$

## 1.2 Solutions and Initial Value Problems

- nth order ODE an equality relating the independent variable to the nth degree derivative of the dependent variable e.g.

$$\underbrace{x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}}_{\text{nth derivative}} + y = x^3$$

Relating — independent var

- General form of an nth order equation with $x$ independent and $y$ dependent.

① $F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right) = 0$

② $\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{n-1} y}{dx^{n-1}}\right)$

1- is preferable for computations

- Explicit Solution
  A function $\phi(x)$ that when substituted for $y$ in equation ① satisfies the equation for all $x$ in the interval $I$ is called an explicit solution to the equation on $I$.

e.g. Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to the linear equation

$$\frac{d^2y}{dx^2} - \frac{2}{x^2} = 0$$

but $\psi(x) = x^3$ is not

Find $\phi'(x)$ and $\phi''(x)$
$\phi'(x) = 2x + x^{-2}$
$\phi''(x) = 2 - 2x^{-3}$

Substitute

$= (2 - 2x^{-3}) - \frac{2}{x^2}(x^2 - x^{-1})$
$= 2 - \frac{2}{x^3} - 2 + \frac{2}{x^3} = 0$

Indeed, it satisfies the eq. equation

e.g. Implicitly Defined solution.

⑤ $y^2 - x^3 + 8 = 0$

Try

⑥ $\frac{dy}{dx} = \frac{3x^2}{2y}$

on the interval $(2, \infty)$

step 1 solve for $y$
$y^2 = x^3 - 8$
$y = \pm\sqrt{x^3 - 8}$

say

$\phi = \sqrt{x^3 - 8}$, $\psi = -\sqrt{x^3 - 8}$

try $\phi(x)$

$\phi' = \frac{3x^2}{2\sqrt{x^3-8}}$

Substitute

$$\underbrace{\frac{3x^2}{2\sqrt{x^3-8}}}_{\phi'} = \frac{3x^2}{2(\underbrace{\sqrt{x^3-8}}_{y})}$$

- Check $\psi(x)$ too.

- Implicit Solution
  A relation $G(x, y) = 0$ is said to be an implicit solution to equation (1) on the interval $I$ if it defines one or more explicit solutions on $I$.

e.g.

Show that
⑦ $x + y + e^{xy} = 0$
is an implicit solution to the non linear equation
⑧ $(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0$

Note! It is not possible to solve for $y$ we will verify using the implicit function theorem – Ask on Details

Do implicit differentiation with respect to

Green

$\frac{d}{dx}(x + y + e^{xy}) = 1 + \frac{dy}{dx} + e^{xy}\left(y + \frac{dy}{dx}x\right)$

$1 + \frac{dy}{dx} + ye^{xy} + \frac{dy}{dx}xe^{yx} = 1 + ye^{xy} + \frac{dy}{dx}(1 + xe^{yx})$

Bingo, Now we have our equation ⑧

Ask! So the reason why we say it satisfies is bc it has the ~~same~~ matching derivative?

- Verifying With implicit diff
  1 take the candidate equation
  2 Differentiate
  3 Manipulate so it looks like the given equation

- Initial value problem
  By an initial value problem of an nth-order differential equation

$$F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right) = 0$$

Find a solution to the differential equation on an interval $I$ that satisfies at $x_0$ the nth initial condition

$y(x_0) = y_0$
$\frac{dy}{dx}(x_0) = y_1$
$\frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1}$

where $x_0 \in I$ and $y_0, y_1, \dots, y_{n-1}$ are given constants

e.g.

Show $\phi(x) = \sin x - \cos x$ is a solution to the initial value problem

⑩ $\frac{d^2y}{dx^2} + y = 0$; $y(0) = -1$, $\frac{dy}{dx}(0) = 1$

First find $\phi'$ and $\phi''$

$\phi'(x) = \cos x + \sin x$

$\phi''(x) = \cos x - \sin x$

Verify by evaluating equations

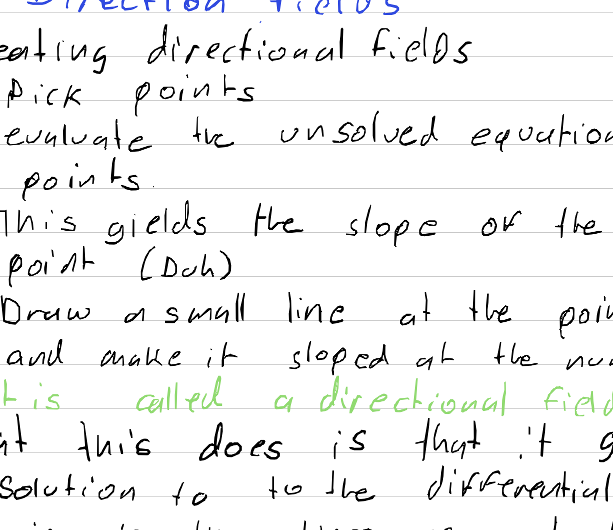
- Existence and Uniqueness of Solution
  Theorem 1. Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

If $f$ and $\partial f/\partial y$ are continuous functions in some rectangle

$$R = \{(x, y) : a < x < b,\ c < y < d\}$$

That contains the point $(x_0, y_0)$, then the initial value problem has a unique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$ where $\delta$ is a positive number.

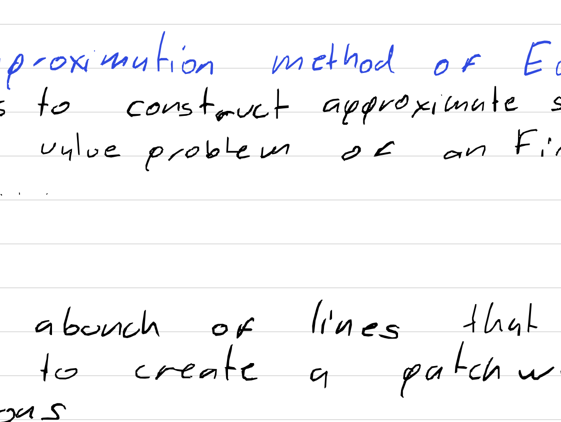



ask! for explanation, im lost

## 1.3 Direction fields

- Creating directional fields
  - Pick points
  - evaluate the unsolved equation at those points
  - This yields the slope of the solution at that point (Duh)
  - Draw a small line at the point chosen and make it sloped at the number found.
- That is called a directional field
- What this does is that it gives the flow of the Solution to to the differential equation.
- Drawing in the lines we get the solutions to different initial valued problems

$$\frac{dy}{dx} = -\frac{y}{x}$$



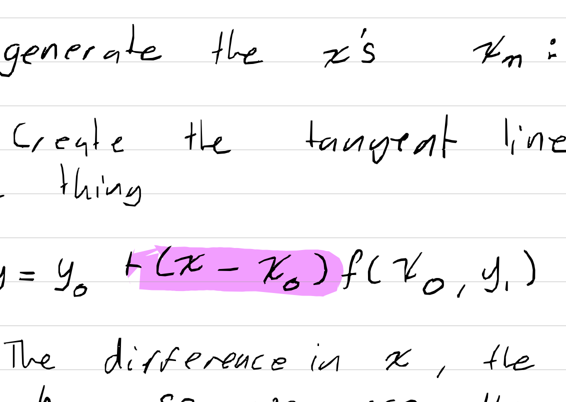

- Some times it might not be too clear how to sketch it right.

- Ask about isoclines, very confussing

## 1.4 The approximation method of Euler

- Objective is to construct approximate solutions to an initial value problem of an First ODE.
- Eulers app... Theory

Construct a bunch of lines that connect to eachother to create a patchwork of linear approximations




1- Start at an initial value given. $(a, b)$
2 - Pick a step, or a leinght in $x$ of the lines. called $h$
3 - generate the $x$'s $x_n : x_0 + nh$ $n \in \mathbb{Z}$
4 - Create the tangent line by using euler's line thing

$$y = y_0 + (x - x_0) f(x_0, y_1)$$

The difference in $x$, the change is indeed $h$ so we use the simpler formula

$$\phi(x_1) \approx y_1 := y_0 + h f(x_0, y_0)$$

Recursive formula

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + h f(x_n, y_1), \quad n \in (\mathbb{Z} \geq 0)$$

- Using Euler's approximation
  1 Find the initial value That will be our $(x_0, y_0)$
  2 Deside on a step: $h$
  3 use the recursive formulas to approximate the solution.

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + h f(x_n, y_1), \quad n \in (\mathbb{Z} \geq 0)$$

- Ask!
  Unsure what we are approximating?
  we pick an interval then we deside on howmany steps the smaller the steps, the better our approximation at that interval

Answered

Say you want to approximate $\phi(2)$ but we only have $\phi(1)$ then we pick a small $h$ to be presice and work our way untill we reach 1.

# 2 First Order Differential Equations

## • 2.4 Exact equations

• How does one calculate the slope of the tangent to a level curve?

— Main gestion

- Take derivative of the equation

$$\frac{d}{dx} F(x,y) = \frac{d}{dx}(c)$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Dumb notation

This solves to be

$$\frac{dy}{dx} = f(x,y) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

small F

The derivative of F

That can be rewritten as

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

where

$$= \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dx$$

$$\frac{dy}{dx} = -\frac{2xy^2+1}{2x^2y}$$

$$\frac{dy}{dx} = -\left(\frac{y}{x} + \frac{1}{2x^2y}\right)$$

$$y\frac{dy}{dx} = -\left(\frac{y^2}{x} + \frac{1}{2x^2}\right)$$

Go over example 1 chap 2.4

• Deff Exact Differential form

# 4.3 Auxilary equations with complex rooths

- Harmonic equation " $y'' + y = 0$ related from musical tones.

has a solution $y_1(t) = \cos t$ and $y_2(t) = \sin t$

- it has complex rooths

$$ar^2 + br + c = 0 \quad (1)$$

when $b^2 - 4ac < 0$

then

$$r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta$$

where

$$\alpha = -\frac{b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

$$e^{\alpha t + i\beta t} = e^{\alpha t} e^{i\beta t}$$

what $e^{i\theta}$ is, use maclauren series

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \dots + \frac{(i\theta)^n}{n!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \dots$$

$$= \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right)}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right)}_{\sin\theta}$$

maclauren series!

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{euler's identity}$$

$$e^{\alpha} e^{i\beta t} = e^{\alpha}(\cos(\beta t) + i\sin(\beta t))$$

going back to ..

$$y(t) = C_1 e^{(\alpha t + \beta t)} + C_2 e^{(\alpha t - \beta t)}$$

$$y(t) = C_1 e^{t}(\cos\beta t + \sin\beta t) + C_2 e^{t}(\cos\beta t - \sin\beta t)$$

Eg. $y'' + 2y' + 2y = 0$ For $y(0) = 0$, $y'(0) = 2$

- auxilary equation

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$$

$\alpha = -1$ The real part

$\beta = 1$

hence

$$y(t) = C_1 e^{-t}(\cos t + \sin t) + iC_2 e^{-t}(\cos t - \sin t)$$

Using int conditions

$$y(t) = -i e^{-t}(\cos t + \sin t) + i e^{-t}(\cos t - \sin t)$$

$$y(t) = 2e^{-t}\sin t$$

- **Lemma 2** Real Solutions derived from complex solutions

Let $z(t) = u(t) + iv(t)$ be a solution to (2) $ay'' + by' + cy = 0$, where $a, b,$ and $c$ are real numbers. Then the real part of $u(t)$ and the imaginary part $v(t)$ are real valued solutions of (2) → (Previous equation)

- Proof by assumption

$$az'' + bz' + cz = 0$$

$$a(u'' + iv'') + b(u' + iv') + c(u + iv) = 0$$

Rearrange

$$(au'' + bu' + cu) + i(av'' + bv' + v) = 0$$

Cleverness: A complex number is zero iff it's real and imaginary parts are zero

$$au'' + bu' + cu = 0 \quad \text{and} \quad av'' + bv' + v' = 0$$

□

When we apply lemma 2 to the solution
$e^{(\alpha + i\beta)t} = e^{\alpha t}\cos\beta t + i e^{\alpha t}\beta t$

we obtain

- Complex conjugate Rooths

If the auxilary equation has complex rooths then two linearly independent solutions

$$e^{\alpha t}\cos\beta t \quad \text{and} \quad e^{\alpha}\sin\beta t$$

and a general solution is

$$(9) \quad y(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$$

where $c_1$ and $c_2$ are arbitrary constants

Ask! Illustrate how we get this much simpler general solution as supposed to the more wordly

$$y(t) = C_1 e^{t}(\cos\beta t + \sin\beta t) + C_2 e^{t}(\cos\beta t - \sin\beta t)$$

e.g. Find general solution
$y'' + 2y' + 4y = 0$

aux

$$r^2 + 2r + 4 = 0$$

$$r = -1 \pm i\sqrt{3} \qquad \alpha = -1 \quad \beta = \sqrt{3}$$

$$y(t) = c_1 e^{-t}\cos\sqrt{3}t + c_2 e^{-t}\sin\sqrt{3}t$$

## 4.4 Nonhomogenous Equations: The method of undetermined coeficients.

- e.g.

$$y_1'' + 3y_1' + 2y_1 = 3t$$

Trying $y(t) = At$ ; $y' = A$ $y'' = 0$

$$0 + 3A + 2At = 3t$$

$$y'' + 3y' + 2y = 3A + 2At$$

Needs to be gone by $A = 0$
and $A$ needs to be $3/2$ to make $3t$

Trying
$$At + B = y$$
$$At = y'$$
$$0 = y''$$

$$2(At + B) + 3A = 2y + 3y' + y''$$
$$2At + 2B + 3A$$

$A$ must be $3/2$
and $2B + 3A = 0$

$$B = -9/4$$

Thus $y(t) = 3/2\, t - 9/4$

- A general solution to problems of this kind

$$ay'' + by' + cy = Ct^m \quad m \in \mathbb{Z}^+$$

we guess a solution of the form

$$y_p(t) = A_m t^m + \dots + A_1 t + A_0$$

- Method of undetermined coefficients involves solving nxn linear equations

- e.g. $y'' + 3y' + 2y = 10e^{3t}$

we guess
$$y = Ae^{3t}$$
$$y' = 3Ae^{3t}$$
$$y'' = 9Ae^{3t}$$

$$9Ae^{3t} + 9Ae^{3t} + 2Ae^{3t}$$
$$20Ae^{3t} = 10e^{3t}$$
$$A = 1/2$$

$$y(t) = \frac{1}{2}e^{3t}$$

- Find a particular solution to

$$y'' + 3y' + 2y = \sin t$$

$A\sin t = y(t)$ won't work

Try
$$A\sin t + B\cos t = y$$
$$A\cos t - B\sin t = y'$$
$$-A\sin t - B\cos t = y''$$

$$-A\sin t - B\cos t + 3A\cos t - 3B\sin t + 2A\sin t + 2B\cos t =$$
$$3A\cos t + B\cos t - 3B\sin t + A\sin t$$
$$= (3A + B)\cos t + (A - 3B)\sin t$$

$$A - 3B = 1$$
$$3A + B = 0 \quad -3 \quad \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 3 & 1 & 0 \end{array}\right]$$

$$1/10 \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 10 & -3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1/10 \\ 0 & 1 & -3/10 \end{array}\right]$$

$$y(t) = \frac{1}{10}\sin t - \frac{3}{10}\cos t$$

- A general guess to this problem

$$ay'' + by' + cy = C\sin\beta t \quad (\text{or } C\cos\beta t)$$

$$y(t) = A\cos\beta t + B\sin\beta t$$

## 4.5 The superposition principle and undetermined coeficients revised.

- Superposition Principle

- Theorem 3 Let $y_1$ be a solution to the differential equation

$$ay'' + by' + cy = f_1(t)$$

and $y_2$ a solution to

$$ay'' + by' + cy = f_2(t)$$

Then for any constants $k_1$ and $k_2$ the function $k_1 y_1 + k_2 y_2$ is a solution to the differential equation

$$ay'' + by' + cy = k_1 f_1(t) + k_2 f_2(t$$

- This helps unite the idea of the particular solution to the idea of the general solution

- Existence and Uniqueness: Nonhomogenous Case

Theorem 4 $\forall a\,(a \neq 0) \vee \forall b, c, t_0, y_0$ and $y_1$, suppose $y_p(t)$ is a particular solution to
$$ay'' + by' + cy = f(t)$$

On an interval $I$ containing $t_0$ and that $y_1(t)$ and $y_2(t)$ are linearly independent Solutions to the homogenous equation

$$ay'' + by' + cy = 0$$

in $I$ then there exists a unique solution $I$ to the initial value problem

$$ay'' + by' cy = f(t)$$

$$y(t_0) = y_0, \quad y'(t_0) = y_1$$

and it's given by the appropiate choise of the constants.

- e.g. given $y_p(t) = t^2$ to

$$y'' - y = 2 - t^2$$

Find the general solution to $y(0) = 1$, $y'(0) = 0$

- homogeneous equation
$$y'' - y = 0$$

$$r^2 - 1 = 0$$
$$r = \pm 1$$

$$C_1 e^{t} + C_2 e^{-t}$$

Combining the general solution to the particular solution

$$y(t) = t^2 + c_1 e^{t} + C_2 e^{-t}$$

To meet the initial conditions

$$y(0) = C_1 + C_2 = 1 \qquad C_1 = C_2$$
$$y'(0) = C_1 - C_2 = 0 \qquad C_2 = \frac{1}{2}$$

$$y(t) = t^2 + \frac{1}{2}(e^{t} + e^{-t})$$
$$= t^2 + \cosh t$$