Lecture Notes 8-9-25

2.2 Separable Equations 2.2

If $f(x,y) = g(x)p(y)$

e.g. $\frac{dy}{dx} = x^2 e^y$ - yes separable

e.g. $\frac{dy}{dx} = 4y^2 - 3y + 1$

$g(x) = 1 \quad p(y) = 4y^2 - 3y + 1$

e.g $\frac{dy}{dx} = t\ln(y^{2t}) + 8t^2 = 2t^2 \ln(y) + 8t^2$

$g(x) = 2t^2 \quad p(y) = \ln(y)$

Solving

$\frac{dy}{dx} = p(y)\, g(x)$ Say $h(y) = \frac{1}{p(y)}$

$h(y)\frac{dy}{dx} = g(x)$

$H'(y) = h(y)$
$G'(x) = g(x)$

$H'(y)\frac{dy}{dx} = G'(x)$

This is the chain rule

$\frac{d}{dx}(H(y)) = \frac{d}{dx}(G(x))$

e.g. $\frac{dx}{dt} = \frac{\sec^2(x)}{1+t^2}$

$\cos^2(x)\, dx = \frac{1}{(1+t^2)}\, dt$

$\int \cos^2(x)\, dx = \int \frac{1}{(1+t^2)}\, dt$

$\frac{1}{2}x + \frac{1}{4}\sin x = \tan^{-1}(t) + C$

Implicit solution No solving for x

e.g. $\frac{dy}{dx} = (1+y^2)\tan(x)$ $\quad y(0) = \sqrt{3}$

$\int \frac{1}{1+y^2}\, dy = \int \tan(x)\, dx$

$\tan^{-1}(y) = \ln|\sec(x)| +$

e.g. $(1/x + 2xy^2)\,dx + (2x^2y - \cos(y))\,dy = 0$

① Verify compatibility

$m = (1/x + 2xy^2)$
$n = (2x^2y - \cos(y))$

② Set $\frac{\partial m}{\partial y} \overset{\text{True}}{=} \frac{\partial n}{\partial x}$

③ $\int \frac{1}{x} + 2xy^2\, dx = \ln|x| + x^2y^2$

$\frac{\partial}{\partial y}\left[\ln|x| + x^2y^2 + g(y)\right] = 2x^2y - \cos(y)$

$g'(y) = -\cos(y)$

$g(x) = -\sin y$

$\ln|x| + x^2y^2 - \sin y = C \qquad y(1) = \pi$

$\ln|1| + (1)^2(\pi)^2 - \sin(\pi) = \pi^2$

$\ln|x| + x^2y^2 - \sin y = \pi^2$ for $x \neq 0$

To solve $\frac{dy}{dx} + P(x)y = Q(x)$

let $\mu(x) = e^{\int P(x)\,dx}$

Multiply original equation by $\mu(x)$

$\mu(x)\frac{dy}{dx} + \mu P(x) y = \mu(x) Q(x)$

$y = \frac{1}{\mu(x)}\left(\int \mu(x) Q(x)\,dx + C\right)$

e.g. $x\frac{dy}{dx} + 2y = x^{-3}$

① Divide by x

$\frac{dy}{dx} + \frac{2}{x}y = x^{-4}$

② Integrate $P(x) = \frac{2}{x}$
- integrating factor

No C

$\mu(x) = e^{2\int \frac{1}{x}dx} = e^{2\ln|x|} = e^{\ln|x^2|} = e^{\ln x^2}$
$= x^2$

③ Multiply by integrating factor (Some times)

$x^2\frac{dy}{dx} + 2xy = x^{-2}$

This is a product rule

$(x^2y)\frac{d}{dx} = x^{-2}$

$x^2y = -\frac{1}{x} + C$

$y = \frac{1}{x^2}\left(-\frac{1}{x} + C\right)$

e.g. $\cos(x)\frac{dy}{dx} + y\sin x = 2x\cos^2(x)$

$y(\pi/4) = -\frac{15\sqrt{2}}{32}\pi^2$

$\frac{dy}{dx} + y\tan(x) = 2x\cos(x)$

$P(x)$ $\tan(x)$

$\mu(x) = e^{\int \tan x\,dx} = e^{\ln|\sec x|} = |\sec x|$

evaluate at initial value

$\sec(\pi/4) = \sqrt{2}$

$\sec(x)\frac{dy}{dx} + y\sec x\tan x = 2x$

$\sec x\, y = x^2 + C$

$y = \cos x(x^2 + C)$

Use initial value

$-\frac{15\sqrt{2}}{32}\pi^2 = \cos(\pi/4)((\pi/4)^2 + C)$

$-\frac{15\sqrt{2}}{32}\pi^2 = \frac{1}{\sqrt{2}}\left(\frac{\pi^2}{16} + C\right)$

$-\frac{15\pi^2}{16} = \frac{\pi^2}{16} + C$

$-\pi^2 = C$

$y = \cos x(x^2 - \pi^2)$
Final Answ

e.g.

$y' + y = \sqrt{1+\cos^2(x)}$, $y(1) = 4$

$\mu(x) = 1$

$\frac{d}{dx}(e^x y) = e^x\sqrt{1+\cos^2 x}$

No closed form anti-derivative

Since we got $y(1) = 1$
we start at 1 to find $y(2)$

$\int_{x=1}^{x=2} e^x y\, d? = \int_1^2 e^x\sqrt{1+\cos^2(x)}\,dx$

$e^x y\Big|_{x=1}^{x=2} = e^2y(2) - e\,y(1) = \int_1^2 e^x\sqrt{1+\cos^2 dx}$

e.g. $\frac{dy}{dx} + y = 0$ $y = e^{-x}$ is a solution

Chapter 2.4

e.g. $(2x+y)\,dx + (x-2y)\,dy = 0$

Find $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = 2x+y \qquad \frac{\partial F}{\partial y} = x-2y$$

① integrate

$$\int 2x+y\,dy = x^2+xy+g(y)$$

② Derivate with respect to y $\frac{\partial}{\partial y}$

$$\frac{\partial}{\partial y}\left[x^2+xy+g(y)\right] = x + g'(y)$$

③ Set that equal to the other derivative

$$x+g'(y) = x-2y$$
$$g'(y) = -2y$$

④ Integrate

$$\int -2y\,dy = -y^2+C$$

C does not matter here

⑤ Put together

$$g(y) = -y^2$$

$$F(x,y) = x^2+xy-y^2 = C$$

• exact equations $\frac{\partial M}{\partial y}(x,y) = \frac{\partial M}{\partial x}(x,y)$

e.g. $\underbrace{(\cos\theta)}_{m}\,dr + \underbrace{(e^{\theta} - r\sin(\theta))}_{n}\,d\theta = 0$

① Exact equations

$$\frac{\partial m}{\partial \theta} = -\sin\theta = \frac{\partial n}{\partial r} = -\sin\theta$$

② $F(r,\theta) = \int m\,dr + g(\theta)$

$$r\cos(\theta) + g(\theta) = F(r,\theta)$$

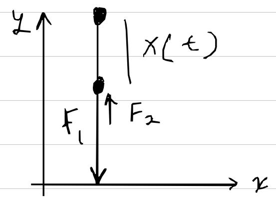
③ $\frac{\partial F}{\partial \theta} = -r\sin\theta + g'(\theta) = e^{\theta} - r\sin\theta$

$$g'(\theta) = e^{\theta}$$

$$g(\theta) = e^{\theta}$$

$r\cos\theta + e^{\theta} =$

Chapter 3.4 Newtonian Objects



Air resistance: F_2

$F_1 = m \cdot g$

$F_2 = -kv$ $\quad k > 0$

BC $\uparrow -$ $\quad \downarrow +$

Where F_2 is air resistance and it is proportional to the velocity v

$$ma = m\frac{dv}{dt} = F_1 + F_2 = mg - kv$$

$$\frac{dv}{dt} = g - \frac{k}{m}v$$

Never mind.

$$\frac{1}{g - \frac{k}{m}v}\, dv = dt$$

$$-\frac{k}{m}\ln\left|g - \frac{k}{m}v\right| = t + C$$

$$\frac{m}{g - kv}\, dV = dt$$

$$-\frac{m}{k}\ln|mg - kv| = t + C$$

$$-\frac{m}{k}\ln(mg - kv) = t + C_1$$

$$\ln(mg - kv) = -\frac{k}{m}t + C_2$$

$$mg - kv = Ce^{\frac{k}{m}t}$$

$$kv = mg - Ce^{-\frac{k}{m}t}$$

$$v = \frac{mg - Ce^{-k/m\, t}}{k} = \frac{mg}{k} - Ce^{-km/m}$$

$$V(0) = V_0$$

$$V = \frac{mg}{k} + \left(V_0 - \frac{mg}{k}\right)e^{-kt/m}$$

$$\frac{dx}{dt} = \frac{mg}{k} + \left(V_0 - \frac{mg}{k}\right)e^{-kt/m}$$

$$x = \int \frac{mg}{k} + \left(V_0 - \frac{mg}{k}\right)e^{-kt/m}$$

$$x(t) = \frac{mg}{k}t - \frac{m}{k}\left(V_0 - \frac{mg}{k}\right)e^{-\frac{kt}{m}} + C$$

$$x(0) = 0$$

$$0 = -\frac{m}{k}\left(V_0 - \frac{mg}{k}\right) + C$$

$$C = \frac{m}{k}\left(V_0 - \frac{mg}{k}\right)$$

$$x(t) = \frac{mg}{k}t - \frac{m}{k}\left(V_0 - \frac{mg}{k}\right)e^{-\frac{kt}{m}} + \frac{m}{k}\left(V_0 - \frac{mg}{k}\right)$$

$$x(t) = \frac{mg}{k}t + \frac{m}{k}\left(V_0 - \frac{mg}{k}\right)\left(1 - e^{-\frac{kt}{m}}\right)$$

e.g. $\frac{dx}{dt} = 1 + t\sin(tx)$; $x(0) = 0$

Approximate $x(1)$ with a tolerance of $\varepsilon = 0.0$

So basically keep shrinking h until the approximation difference is less than $\varepsilon = n$

3.7

- taylor Method and Runge - Kutta method

let $\phi_n(x)$ be the exact solution to

$\phi_n' = f(x, \phi_n)$; $\phi_n(x_n) = y_n$

The taylor series expansion for $\phi_n(x)$ about x_n

$$\phi_n(x) = \phi_n(x_n) + h\phi_n'(x_n) + \frac{h^2}{2!}\phi''(x_n) + \dots$$

where $h = x - x_n$

$$\phi_n(x) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^n(x_n)$$

$$\phi_n(x) = \phi_n(x_n) + h\phi_n'(x_n)$$ Basically Euler's method

let $y' = f(x,y)$ &

then $y'' = \frac{\partial f}{\partial x}\cancel{\frac{dx}{dx}} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x,y) = f_2(x,y)$$

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2!} f_2(x,y)$$

e.g.

$$y'' - 5y' + 6y = 0$$

characteristic equation

$$r^2 - 5 + 6y = 0$$
$$r = 2, 3$$

$$e^{2t}, e^{3t}$$

e^{2t} and e^{3t} they both satisfy $y'' - 5y' + 6y = 0$
so $e^{2t} + e^{3t}$ will also satisfy because we separate derivatives so general solution is

$$y = c_1 e^{2t} + c_3 e^{2t}$$

we need two initial values

e.g. $y'' - 4y' - 5y = 0$ $\quad y(-1) = 3,\ y'(-1) = 9$

$$r^2 - 4r - 5 = 0$$
$$r = \{-1, 5\}$$

$$y = C_1 e^{-t} + C_2 e^{5t}$$

use $y(-1) = 3$

$$3 = C_1 e + C_2 e^{-5}$$

$$y' = -C_1 e^{-t} + 5C_2 e^{5t}$$

$$9 = -C_1 e + 5C_2 e^{-5}$$

$$\begin{cases} 3 = C_1 e + C_2 e^{-5} \\ 9 = -C_1 e + 5C_2 e^{-5} \end{cases}$$

$$12 = 6C_2 e^{-5} \qquad 3 = C_1 e + 2$$

$$2 = C_2 e^{-5} \qquad e^{-1} = C_1$$

$$2e^5 = C_2$$

$$y = e^{-(t+1)} + 5e^{5t+5}$$

- Uniqueness and existance

If $y_1(t)$ and $y_2(t)$ are any solutions to $ay'' + by' + cy = 0$ that are linearly independent on $(-\infty, \infty)$, then unique constants, C_1 and C_2, can be found so that $C_1 y_1(t) + C_2 y_2(t)$ satisfies $ay'' + by' + cy =$, $y(t_0) = y_0$, $y'(t_0) = y_1$.

Lemma: for any real numbers a, b, c $\forall (a, b, c \in \mathbb{R})$ if $y_1(t)$ and $y_2(t)$ are any two solutions to $ay'' + by' + cy = 0$ on $(-\infty, \infty)$ and $y_1(t_1)\, y_2'(t_1) - y_1'(t_1)\, y_2(t_1) = 0$ at some point t_1 then y_1 and y_2 are linearlly independent $(-\infty, \infty)$

e.g. $y'' + 8y' + 16y = 0$

$$r^2 + 8r + 16 = 0$$

$r = \{-4\}$ but $y = e^{-4t} + e^{-4t}$ won't work
bc Not linearl'indi..

Just add a t

$$y = e^{-4t} + te^{-4t}$$

$$y_2 = te^{-4t}$$
$$y' = e^{-4t} - 4te^{-4t} = e^{-4t}(1 - 4t)$$
$$y'' = e^{-4t}(16t - 8)$$

when we plug in to $y_2'' + 8y' + 16y$ we get 0

$$y = C_1 e^{-4t} + C_2 t e^{-4t}$$

$$\boxed{y = e^{-4t}(C_1 + C_2 t)}$$

Lecture Notes

- Cauchy Euler

$at^2y''(t) + bty'(t) + cy = f(t)$

- e.g. $t^2 \frac{d^2y}{dt^2} + 2t\frac{dy}{dt} + 6y = 0$

Let $y = t^r$ $t > 0$

$y' = rt^{r-1}$
$y'' = r(r-1)t^{r-2}$

$r(r-1)t^r + 2rt^r - 6t^r = 0$
$t^r(r^2 - r + 2r - 6) = 0$
$t^r(r^2 + r - 6) = 0$

$r^2 + r - 6 = 0$
$r = 3, -2$

$y_1(t) = t^{-2}$, $y_2(t) = t^3$

- What if $r = \alpha + i\beta$

$t^{\alpha + i\beta} = t^\alpha \cdot t^{i\beta} = t^\alpha e^{i\beta\ln(t)} =$

$t^\alpha(\cos(\beta\ln t) + i\sin(\beta\ln t))$

$y_1 = t^\alpha\cos(\beta\ln(t))$, $y_2 = t^\alpha\sin(\beta\ln(t))$

- repeated Roots

$y = t^r$, $y_2 = t^r\ln(t)$

- e.g. $t^2y'' - 3ty' + 4y = 0$

$y = t^r$
$y' = rt^{r-1}$
$y'' = r(r-1)t^{r-2}$

$r(r-1)t - 3rt^r + 4t^r = 0$

$t^r(r^2 - 4r + 4) = 0$

$r = 2$

$y(t) = t^2$ $y_2(t) = t^2\ln(t)$

- e.g. $t^2y'' + 5ty' + 12y = 0$ $t > 0$

$r^2 - r + 5r + 12 = 0$
$r^2 + 4r + 12 = 0$

$r^2 + 4r = -12$

$r^2 + 4r + 4 = -8$
$(r+2)^2 = -8$
$r + 2 = \pm i2\sqrt{2}$
$r = 2 \pm 2i\sqrt{2}$

$y_1 = t^{-2}\cos(2\sqrt{2}\ln(t))$

$y_2 = t^{-2}\sin(2\sqrt{2}\ln(t))$

- Lemma If y_1 and y_2 are solutions to

$y'' + p(t)y' + q(t)y = 0$

on an interval I where P and Q are continuous
is the wronskian

$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = 0$

on the interval I then y_1 and y_2 are linearly Dependent on I

- Theorem If y_1 and y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$ that are linearly dependent on an interval I containing t_0, then unique constants C_1 and C_2 can always be found so that $C_1y_1 + C_2y_2$ satisfies
$y(t_0) = y_0$ $y'(t_0) = y_0'$

- e.g. $t^2y'' + ty' + 9y = -\tan(3\ln(3))$

$r^2 + 9 = 0$
$r^2 = -9$
$r = \pm 3i$

$y_1(t) = \cos(3\ln t)$

$y_2(t) = \sin(3\ln t)$

- e.g.

$y'' + 4y = 5t^2e^t$ • Variation

$y(t) = (At^2 + Bt + C)e^t$
$y'(t) = e^t(At^2 + Bt + C) + e^t(2At + B)$
$y'(t) = e^t(At^2 + Bt + C + 2At + B)$

$y'(t) = e^t(At^2 + 2At + Bt + B + C)$

$y''(t) = e^t(At^2 + 2At + Bt + B + C)$

$+ e^t(2At + 2A + B)$

$y''(t) = e^t(At^2 + 4At + 2A + Bt + 2B + C)$

5.3 section

$y'' + y' \frac{1}{1+t^2} - y \frac{1}{21+t^2}$ $y(0) = 1$ $y'(0) = -1$

$h = 0.25$ on $[0, 1]$

$\overline{X}' = \langle x_2, \frac{1}{1+t^2}(x_1 - x_2) \rangle = \bar{F}$

6.1

8/12/23

e.g. $\mathcal{L}\{t^2 \cos(bt)\} = ?$

Recall $\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}$

$$\mathcal{L}\{t^2 \cos(bt)\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+b^2}\right)$$

$$= \frac{2s(s^2-3b^2)}{(s^2+b^2)^3}$$

Inverse Laplace Transfor 7.4

e.g. $y'' - y = -t \quad y(0) = \quad y'(0) =$

$$\mathcal{L}\{y'' - y\} = \mathcal{L}\{-t\}$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = -\frac{1}{s^2}$$

$$\mathcal{L}\{y''\} = s^2 \mathcal{L}\{y\} - \underbrace{s\,y(0)}_{0} - \underbrace{y'(0)}_{1}$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\}\ s^2 - 1$$

Note $Y = \mathcal{L}\{y\}$

$$s^2 Y - 1\ Y = -\frac{1}{s^2}$$

$$Y = (1 - 1/s^2)/(s^2-1) = \frac{\cancel{s^2-1}}{(\cancel{s^2-1})\,s^2} = \frac{1}{s^2}$$

$$\mathcal{L}\{y\} = \frac{1}{s^2} = t \qquad \text{bc } t^n = \frac{n!}{s^n}$$

Other solution could be piecewise continuous functions ... we want continuous functions

e.g $F(s) = \frac{6}{(s-1)^4}$

$$\mathcal{L}\{t^n e^{\alpha t}\} = \frac{n!}{(s-\alpha)^{n+1}}$$

In this case $n = 3$ and $\alpha = 1$

$$\mathcal{L}^{-1}\left\{\frac{6}{(s-1)^4}\right\} = t^3 e^t$$

e.g $F(s) = \frac{s+1}{s^2+2s+10}$ $\quad a = -1$, $b = 3$

$$\mathcal{L}^{-1}\{F\} = e^{-t}\cos(3t)$$

e.g. $F(s) = \frac{s^2-26s-47}{(s-1)(s+2)(s+5)}$

$$y'' - 6y' + 15y = 2\sin(3t) \quad y(0) = -1, \; y'(0) = -4$$

$$s^2 Y(s) - sY(0) - y'(0) - 6(sY(s) - y(0)) + 15Y(s)$$
$$= \frac{3}{s^2+9} \cdot 2$$

$$(s^2 - 6s + 15)Y + s + 4 - 6 = \frac{6}{s^2+9}$$

$$Y(s) = \frac{-s^3 - 2s^2 - 9s + 24}{(s^2+9)(s^2-6s+15)} = \frac{As + B}{s^2+9} + \frac{Cs + D}{s^2 - 6s + 15}$$

$$x_1' = f_1(t, x_1, \ldots x_m)$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \vec{f}(x\cdots) = \begin{bmatrix} f_1 \\ \\ f_n \end{bmatrix}$$

e.g. $y'' + \frac{y'}{1+t^2} - \frac{y}{1+t^2} = 0 \quad y(0) = 1 \quad y'(0) = -1$

$h = .25$ on $(0, 1)$

$$y'' - 3ty' - 6y = 2\cos t$$

$$x_1 = y$$

$$x_1' = y' = x_2$$

$$x_2' - 3t x_2 - 6x_1 = 2\cos t$$

$$x_2'(0) = -3t$$

$$x_2' = 3\sin(t - x_1) - 6x_1^2$$

$$x'' + 5y - 4x' = t$$

$$y'' - 2x + 3y = -2;$$

$$x_2' + 5x_3 - 4x_2 = t$$

$$x_2' = t - 5x_3 + 4x_2$$

$$x_4' - 2x_1 + 3x_3 = -2$$

$$x_4' = 2x_1 - 3x_3 - 2$$

$$7x_2' + 2x_1 - 3x_3 = 0$$

$$x_2' = \frac{1}{7}(3x_3 - 2x_1)$$

$$x_4' = \frac{1}{8}(10x_1 - 6x_3)$$

$x(x+1)y''' - 3xy' + y = 0$

$y(-1/2) = 1, \quad y'(-1/2) = 0, \quad y''(-1/2) = 0$

Find the largest interval (a,b) where theres a guaranteed unque solution.

$$y''' - \frac{3y'x}{(x+1)x} + \frac{y}{x(x+1)} = 0$$

from $(-1, 0)$