Exam 1 Practice

① Slope field for $\frac{dy}{dx} = (y^2 - y - 2)(1-y)^2$
Describe the behavior of y as $x \to \infty$ for given initial conditions

a) $y(0) = -1$ $\quad y \to 1$
b) $y(0) = 1.5$ $\quad y \to \infty$
c) $y(0) = 2$ $\quad y \to \infty$
d) $y(0) = 3$ $\quad y \to \infty$

Connect the dots (Tangent lines)

② Euler's method Round to 3 desimal places

a) $y' = -\frac{x}{y}, y(0) = 4$ $\quad x = \{0.1, 0.2, \ldots, 0.5\}$

Eulers Method

$$x_{n+1} = x_n + h$$
$$y_{n+1} = y_0 + h f(x_n, y_n)$$

$h = 0.1$
$y_n = 4$
$x_n = 0$

$x_0 = 0 \quad y_0 = 4 + 0.1\left(-\frac{0}{4}\right) = 4$

$x_1 = 0.1 \quad y_1 = 4 + 0.1\left(-\frac{0.1}{4}\right) = 3.9975$
≈ 3.998

$x_2 = 0.2 \quad y_2 = 3.9975 + 0.1\left(-\frac{0.2}{3.9975}\right)$
$= 3.992496873$
≈ 3.992

$x_3 = 0.3 \quad y_3 = 3.992 + 0.1\left(-\frac{0.3}{3.992}\right)$
$= 3.984982778$
≈ 3.985

$x_4 = 0.4 \quad y_4 = 3.985 + 0.1\left(-\frac{0.4}{3.985}\right)$
$= 3.974945094$
≈ 3.975

$x_5 = 0.5 \quad y_5 = 3.975 + 0.1\left(-\frac{0.5}{3.975}\right)$
$= 3.962366304$
≈ 3.962

Aswer

x_n	y_n
$x_0 = 0$	$y_0 = 4$
$x_1 = 0.1$	$y_1 = 3.998$
$x_2 = 0.2$	$y_2 = 3.992$
$x_3 = 0.3$	$y_3 = 3.985$
$x_4 = 0.4$	$y_4 = 3.975$
$x_5 = 0.5$	$y_5 = 3.962$

b) $y' = x - y^2, \quad y(1) = 0$ at $x = 1.1, 1.2, 1.3, 1.4, 1.5$
$x - y^2$

x_n	y_n	$f(x_n, y_n)$	$y_{n+1} = y_n + 0.1 f(x_n, y_n)$
1	0	1	0.1
1.1	0.1		0.209
1.2	0.209		0.3246319
1.3	0.325		0.444093313
1.4	0.444		0.5643714259
1.5	0.564		0.6

Eulers Method
① state the variables
x_0, y_0, h
② Use a table and calculator

$x_n \quad y_n \quad y_{n+1} = y_n + h f(x_n, y_n)$

③ Solve the following initial value problems.

$\frac{dy}{dx} = x^2(1-y), \quad y(0) = 3$

$(1-y)\,dy = x^2\,dx$

$\int 1 - y\,dy = \int x^2\,dx$

$y - \frac{1}{2}y^2 = \frac{1}{3}x^3 + C$

$3 - \frac{1}{2}(3)^2 = \frac{1}{3}(0)^3 + C$

$3 - \frac{1}{2}9 = C$

$\frac{6-9}{2} = -\frac{3}{2}$

$y - \frac{1}{2}y^2 = \frac{1}{3}x^3 - 3/2$

b) $\frac{1}{t}\frac{dy}{dt} = 2\cos^2 y, \quad y(0) = \pi/4$

$t\frac{dt}{dy} = \frac{1}{2\cos^2 y}$

$t\,dt = \frac{1}{2\cos^2 y}\,dy$

$\int t\,dt = 2\int \sec^2 y\,dy$

$C + \frac{1}{2}t^2 = 2\tan y$

$C = 2$

$2 + \frac{1}{2}t^2 = 2\tan y$

c) $\frac{1}{\theta}\frac{dy}{d\theta} = \frac{y\sin\theta}{y^2+1} \quad y(\pi) = 1$

$\frac{1}{\theta\sin\theta}\frac{dy}{d\theta} = \frac{y}{y^2+1}$

$\int \theta\sin\theta\,d\theta = \int y + \frac{1}{y}\,dy$

$-\theta\cos\theta + \sin\theta + C = \frac{1}{2}y^2 + \ln|y|$

$-1 + C = \frac{1}{2}$

$C = 3/2$

$\sin\theta - \theta\cos\theta + 3/2 = 1/2\, y^2 \ln|y$

d) $\frac{dy}{dx} + 4y - e^{-x} = 0 \quad y(0) = 4/3$

$\frac{dy}{dx} + 4y = e^{-x}$

$P(x) = 4 \quad \int 4x\,dx = 4x \quad \mu(x) = e^{4x}$

$e^{4x}\frac{dy}{dx} + 4ye^{4x} = e^{3x}$

$\frac{d}{dy}[e^{4x}y] = \int e^{3x}\,dx$

$e^{4x}y = \frac{1}{3}e^{3x} + C$

$y = \frac{1}{3}e^{-x} + \frac{C}{e^{4x}}$

$4/3 = \frac{1}{3}\cancel{e^{-(0)}}^{1} + C\,\cancel{\frac{1}{e^{4(0)}}}^{1}$

$4/3 = 1/3 + C$

$1 = C$

$y = \frac{1}{3}e^{-x} + e^{-4x}$

e) $x\frac{dy}{dx} + 3(y + x^2) = \frac{\sin x}{x}; \quad y(1) = \sin(1) - \cos(1)$

$x\frac{dy}{dx} = \frac{\sin x}{x} - 3y - 3x^2$

$\frac{dy}{dx} = \frac{\sin x}{x^2} - \frac{3y}{x} - 3x$

$\frac{dy}{dx} + 3\frac{y}{x} = \frac{\sin(x)}{x^2} - 3x$

$P(x) = 3/x \quad 3\int 1/x\,dx = 3\ln|x|$

$\mu = e^{3\ln|x|} = x^3$

$x^3\frac{dy}{dx} + 3x^2y = x\sin(x) - 3x^4$

$yx^3 = \int x\sin(x) - 3x^4\,dx$

$yx^3 = -x\cos x + \sin x - \frac{3}{5}x^5 + C$

$yx^3 = \sin x - x\cos x - \frac{3}{5}x^5 + C$

$y = \frac{\sin x}{x^3} - \frac{\cos x}{x^2} - \frac{3}{5}x^2 + \frac{C}{x^3}$

$\cancel{\sin(1)} - \cancel{\cos(1)} = \cancel{\sin(1)} - \cancel{\cos(1)} - \frac{3}{5} + C$

$3/5 = C$

$y = \frac{\sin x}{x^3} - \frac{\cos x}{x^2} - \frac{3}{5}x^2 + \frac{3}{5x^3}$

f) $\left(\frac{1}{x} + 2y^2x\right)dx + (2yx^2 - \cos(y))\,dy = 0$
$y(1) = \pi$

- Test for exactness

$\frac{\partial}{\partial y}\left(\frac{1}{x} + 2y^2x\right) = 4yx$

$\frac{\partial}{\partial x}(2yx^2 - \cos y) = 4yx$

Exact

$\int \frac{1}{x} + 2y^2x\,dx$

$= \ln|x| + y^2x^2 + g(y)$

$\frac{\partial}{\partial y}[\ln|x| + y^2x^2 + g(y)] =$

$\cancel{2yx^2} + g'(y) = \cancel{2yx^2} - \cos(y)$

$g'(y) = -\cos(y)$

$g(y) = -\sin(y) + C$

$\ln|x| + y^2x^2 - \sin(y) = C$

$0 + (\pi)^2(1)^2 - \sin(\pi) = C$

$\pi^2 - 1 = C$

$\ln|x| + y^2x^2 - \sin y = \pi^2$

④ A nitric acid solution flows at a constant rate of 6 $\frac{\text{Liters}}{\text{min}}$ into a tank that initially held 200L of a 0.5% nitric acid solution. The solution inside the tank is kept well stired and flows out of the tank at a rate of 8L/min. If the solution entering the tank is 20% nitric acid, determine the volume of nitric acid in the tank after t minutes. When will the percentage of nitric acid reach 10%?

Input Rate $= (6)(0.2) = 1.2$
Output Rate $= 8\left(\frac{x}{100-2t}\right) = \frac{4x}{50-t}$

$\frac{dx}{dt} = 1.2 - \frac{4x}{50-t}$

$\frac{dx}{dt} + \frac{4}{50-t}x = 1.2$

$P(t) = 4\frac{1}{50-t} \to 4\int \frac{1}{50-t}\,dt = -4\ln|50-t|$

$e^{-4\ln|50-t|} = (50-t)^{-4} = \mu(t)$

$(50-t)^{-4}x = \int 1.2(50-t)^{-4}\,dt$

$(50-t)^{-4}x = 0.4(50-t)^{-3} + C$

$x(0) = 1$

$(50)^{-4}(1) = 0.4(50)^{-3} + C$

$0.05 = C$

$x(t) = 0.4(50-t) + \frac{0.05}{(50-t)^{-4}}$

⑤ Improved Euler's method

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n))]$$

a) $\frac{dy}{dx} = -\frac{x}{y}; \quad y(0) = 4$ at $x = 0.1, 0.2, \ldots, 0.5$

x_n	y_n	$f(x_n, y_n)$	$y_n + hf(x_n, y_n)$	$f(x+h, y_n + hf(x_n, y_n))$
0.1	4	-0.025	3.9975	-0.0500312654
0.2	3.9962	-0.05005	3.9911	-0.0751654522
0.3	3.9899			

Exam # 2 practice

① $y'' - 4y' + 3y = 0 \quad , y(0) = 1 \quad y'(0) = \frac{1}{3}$

$r^2 - 4r + 3 = 0$

$r = 3, 1$

$y(t) = C_1 e^{3t} + C_2 e^{t}$

$y'(t) = 3C_1 e^{3t} + C_2 e^{t}$

$C_1 + C_2 = 1$

$3C_1 + C_2 = 1/3$

$C_1 = -\frac{1}{3}$

$C_2 = 4/3$

$y(t) = -\frac{1}{3} e^{3t} + 4/3 e^{t}$

② $y'' - 8y' + 16y = 0 \quad y(0) = 1 \quad y'(0) = -1$

$r^2 - 8r + 16 = 0$

$r = 4$

$y(t) = C_1 e^{4t} + C_2 t e^{4t}$

$y'(t) = 4C_1 e^{4t} + C_2(1 + 4t) e^{4t}$

$C_1 + \cancel{C_2(0)} = 1$

$4C_1 + C_2(1 + 0) = -1$

$4 + C_2 = -1$

$C_2 = -5$

$C_1 = 1$

$C_2 = -5$

$y(t) = e^{4t} - 5 t e^{4t}$

③ $y'' - 2y' + 5y = 0 \quad y(0) = 1, \; y'(0) = 2$

$r^2 - 2r + 5 = 0$

$r^2 - 2r = -5$ $\qquad \sin(2t)$

$r^2 - 2r + 1 = -4$

$(r - 1)^2 = -4$

$r - 1 = \pm 2i$

$r = 1 \pm 2i$

$y = e^{t}(C_1 \cos 2t + C_2 \sin 2t)$

$y' = e^{t}(C_1 \cos 2t + C_2 \sin 2t$
$\quad -2C_1 \sin 2t + 2C_2 \cos 2t)$
$= e^{t}((C_1 + 2C_2) \cos 2t + (C_2 - 2C_1) \sin 2t)$

$C_1 = 1$

$C_1 + 2C_2 = 2$

$2C_2 = 1$

$C_2 = \frac{1}{2}$

$y(t) = e^{t}(\cos 2t + \frac{1}{2} \sin 2t)$

④ a) $y'' + 9y = 4t^3 + \sin(3t)$

$r^2 + 9 = 0$

$r = \pm 3i \rightarrow C_1 \cos 3t + C_2 \sin 3t$

$y_p = (A_0 + A_1 t + A_2 t^2 + A_3 t^3) + t(B_0 \cos 3t + B_1 \sin 3t)$

b) $y'' - 4y' + 5y = e^{5t} + t \sin 3t - \cos(3t)$

$r^2 - 4r + 5 = 0$

$r^2 - 4r = -5$ $\qquad e^{2t}(C_1 \sin t + C_2 \cos t)$

$r^2 - 4r + 4 = -1$

$(r - 2)^2 = -1$

$r - 2 = \pm i$

$r = 2 \pm i$

$y_p = Ae^{5t} + (B_0 + B_1 t)(C_1 \sin 3t + C_2 \cos 3t)$
$+ D_0 \cos 3t + D_1 \sin 3t)$

⑤ $y'' - y' - 2y = \cos(x) - \sin(2x)$

$y(0) = -7/20$

$y'(0) = \frac{1}{5}$

$y'' - y' - 2y = 0 \quad y_h = C_1 e^{-x} + C_2 e^{2x}$

$y = -1, 2$

$y_p = A\cos(x) + B\sin(x) + C\cos(2x) + D\sin(2x)$

$y'_p = -A\sin(x) + B\cos(x) - 2C\sin(2x) + 2D\cos(2x)$

$y''_p = -A\cos(x) - B\sin(x) - 4C\cos(2x) - 4D\sin(2x)$

$-A\cos x - B\sin x - 4C\cos 2x - 4D\sin 2x$
$+A\sin x - B\cos x + 2C\sin 2x - 2D\cos 2x$
$-2A\cos x - 2B\sin x - 2C\cos 2x - 2D\sin 2x$

$(-3A - B)\cos x + (-3B + A)\sin x$

$(-6C - 2D)\cos 2x + (-6D + 2C)\sin 2x =$

$\cos(x) - \sin(2x)$

$-3A - B = 1 \qquad A = -\frac{3}{10} \qquad -6C - 2D = -1$

$A - 3B = 0 \qquad B = -\frac{1}{10} \qquad 2C - 6D = 0$

$C = \frac{3}{20}$

$D = \frac{1}{20}$

$y = -\frac{3}{10}\cos x - \frac{1}{10}\sin x + \frac{3}{20}\cos 2x + \frac{1}{20}\sin 2x + C_1 e^{-x} + C_2 e^{2x}$

$y(0) = -\frac{3}{10} + \frac{3}{20} + C_1 + C_2 = -\frac{7}{20}$

$y' = \frac{3}{10}\sin x - \frac{1}{10}\cos x - \frac{6}{20}\sin 2x + \frac{1}{10}\cos 2x - C_1 e^{-x} + 2C_2 e^{2x}$

$y'(0) = \cancel{-\frac{1}{10} + \frac{1}{10}} - C_1 + 2C_2 = \frac{1}{5}$

$C_1 + C_2 = -\frac{1}{5} \qquad C_1 = -\frac{1}{5}$

$-C_1 + 2C_2 = \frac{1}{5} \qquad C_2 = 0$

$y = -\frac{3}{10}\cos x - \frac{1}{10}\sin x + \frac{3}{20}\cos 2x + \frac{1}{20}\sin 2x - \frac{1}{5}e^{-x}$

6 $v'' + 4v = \sec^4(2t)$

$r^2 + 4 = 0$

$r = \pm 2i$

$y_1 = \cos 2t \qquad y_2 = \sin 2t$

$y_1' = -2\sin 2t \qquad y_2' = 2\cos 2t$

$\begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} \qquad 2\cos^2 2t + 2\sin^2 2t = \boxed{2}$

$v_1 = -\int \frac{\sec^4 2t \cdot \sin 2t}{2} dt = -\frac{1}{2}\int \frac{\sin 2t}{\cos^4 2t} dt \qquad u = \cos 2t, \; du = 2\cos 2t \, dt$

$= -\frac{1}{4}\int \frac{1}{u^4} du = \frac{1}{12}u^{-3} = \frac{1}{12}\sec^3 2t$

$v_2 = \int \frac{\sec^4 \cdot \cos 2t}{2} du = \frac{1}{2}\int \sec^3 2t \, dt$

$\int \sec^2 \cdot \sec dt$

$u = \sec 2t \qquad dv = \sec^2 2t$

$du = 2\sec 2t \tan 2t \quad v = \frac{1}{2}\tan 2t$

$I = \frac{1}{2}[\frac{1}{2}\tan 2t \sec 2t - \int \tan^2 2t \sec 2t \, dt]$

$I = \frac{1}{2}[\frac{1}{2}\tan 2t \sec 2t - \int (1 - \sec^2 2t) \sec 2t \, dt]$

$I = \frac{1}{2}[\frac{1}{2}\tan 2t \sec 2t - \int \sec 2t + \int \cancel{\sec^3 2t} \, dt]$

$I = \frac{1}{2}[\frac{1}{2}\tan 2t \sec 2t + \frac{1}{2}\ln|\sec 2t + \tan 2t| + I]$

$\frac{1}{2}I = \frac{1}{4}\tan 2t \sec 2t + \frac{1}{4}\ln|\sec 2t + \tan 2t|$

$I = \frac{1}{2}\tan 2t \sec 2t + \frac{1}{2}\ln|\sec 2t + \tan 2t|$

$y = C_1\cos 2t + C_2\sin 2t + \frac{1}{12}\sec^2 2t + \frac{1}{2}\sin 2t(\tan 2t \ldots$

⑦ $t^2 y'' - 4ty + 4y = 0 \quad y(1) = -2 \; ; \; y'(1) = 11$

$y = t^r \qquad y' = r t^{r-1} \qquad y'' = r(r-1)t^{r-2}$

$t^2(r(r-1)t^{r-2}) - 4tr t^{r-1} + 4t^r = 0$

$t^r(\cancel{t^2 \cdot t^{-2}}(r^2 - r) - 4r\cancel{t t^{-1}} + 4) = 0$

$t^r(r^2 - r - 4r + 4) = 0$

$t^r(r^2 - 5r + 4) = 0$

$r^2 - 5r + 4 = 0$

$r = 1, 4$

$y = C_1 t + C_2 t^4$

$y' = C_1 + 4C_2 t^3$

$y(1) = C_1 + C_2 = -2 \qquad C_1 =$

$y'(1) = C_1 + 4C_2 = 11 \qquad C_2 =$