Exam 3 — Practice

1. Use the elimination method to find a general solution to the following linear systems. Differentiation is with respect to t.

   (a) $\begin{cases} x' + 2y = 0 \\ x' - \ \ y' = 0 \end{cases}$

   (b) $\begin{cases} (D+1)[u] - \quad (D-1)[v] = e^t \\ (D-1)[u] - (2D+1)[v] = 5 \end{cases}$

2. Use the Wronskian to verify that the given functions form a fundamental set for the given differential equation and find a general solution.

   (a) $y''' + 2y'' - 11y' - 12y = 0, \left\{e^{3x}, e^{-x}, e^{-4x}\right\}$

   (b) $x^2y''' - 3x^2y'' + 6xy' - 6y = 0, \left\{x, x^2, x^3\right\}$

3. Find a general solution to the given homogeneous equation.

   (a) $(D-1)^2(D+3)(D^2+2D+5)^2[y] = 0$

   (b) $(D+4)(D-3)(D+2)^3(D^2+4D+5)^2D^5[y] = 0$

4. Use the annihilator method to determine the form of a particular solution for the given equation.

   (a) $u'' - 5u' + 6u = \cos(2x) + 1$

   (b) $y'' + 2y' + 2y = e^{-x}\cos(x) + x^2$

5. Solve the initial value problem $y''' + 2y'' - 9y' - 18y = -18x^2 - 18x + 22$, $y(0) = -2$, $y'(0) = -8$, and $y''(0) = -12$

6. Use the definition of the Laplace transform to determine the Laplace transform of the given functions. (So you're going to need to integrate)

   (a) $e^{2t}\cos(3t)$

   (b) te^{3t}

7. Use the Laplace transform table and linearity to determine the following transforms.

   (a) $\mathcal{L}\left\{6e^{-3t} - t^2 + 2t - 8\right\}$

   (b) $\mathcal{L}\left\{t^4e^{5t} - e^t\cos(t\sqrt{7})\right\}$

8. Determine $\mathcal{L}^{-1}\{F\}$.

   (a) $F(s) = \dfrac{6s^2 - 13s + 2}{s(s-1)(s-6)}$

   (b) $F(s) = \dfrac{7s^2 - 41s + 84}{(s-1)(s^2-4s+13)}$

9. Use the method of Laplace transforms to solve the following initial value problems.

   (a) $y'' + 6y' + 9y = 0$, $y(0) = -1$, $y'(0) = 6$

   (b) $z'' + 5z' - 6z = 21e^{t-1}$, $z(1) = -1$, $z'(1) = 9$

Answer Key

1. (a) $x = c_1 + c_2^{-2t}$, $y = c_2 e^{-2t}$

   (b) $u = c_1 - \frac{1}{2} c_2 e^{-t} + \frac{1}{2} e^{t} + \frac{5}{3} t$,

   $v = c_1 + c_2 e^{-t} + \frac{5}{3} t$

2. (a) $c_1 e^{3x} + c_2 e^{-x} + c_3 e^{-4x}$

   (b) $c_1 x + c_2 x^2 + c_3 x^3$

3. (a) $c_1 e^{x} + c_2 x e^{x} + c_3 e^{-3x} + (c_4 x + c_5) e^{-x} \cos(2x) + (c_6 x + c_7) e^{-x} \sin(2x)$

①

$$\begin{cases} x' + 2y = 0 \\ x' - y' = 0 \end{cases}$$

$$\begin{cases} Dx + 2y = 0 \\ Dx - Dy = 0 \end{cases}$$

$$\begin{cases} Dx + 2y = 0 \\ 0 - (D-2)y = 0 \end{cases}$$

$y' - 2 = 0$

$Ce^{-2t} = y$

$$\begin{cases} Dx + 2y = 0 \\ Dx - Dy = 0 \end{cases}$$

$D(Dx + 2y) = 0$

$2(Dx - Dy) = 0$

$D^2x + 2Dy = 0$

$2Dx - 2Dy = 0$

$D^2x + 2Dx = (D^2 - 2D)[x] = 0$

$x'' + 2x' = 0$

$r^2 + 2r = 0$

$r(r+2) = 0$

$C_1 e^0 + C_2 e^{-2t}$

① $y(t) = C_1 e^{-2t}$; $x(t) = C_2 e^{-2t} + C_3$

ⓑ $$\begin{cases} (D+1)[u] - (D-1)[v] = e^t \\ (D-1)[u] - (2D+1)[v] = 5 \end{cases}$$

$$\begin{cases} (D+1)u - (D-1)v = e^t \\ (D-1)u - (2D+1)v = 5 \end{cases}$$

$$-\begin{cases} (2D+1)(D+1)u - (2D+1)(D-1)v = (2D+1)e^t \\ (D-1)^2 u - (2D+1)(D-1)v = (D-1)5 \end{cases}$$

$[(2D+1)(D+1) - (D-1)^2]u = (2D+1)e^t - (D-1)5$

$[2D^2 + 3D + 1 - D^2 + 2D - 1]u = 2e^t + e^t + 5$

$[D^2 + 5D]u = 3e^t + 5$

$r^2 + 5r = 0$

$r(r+5) = 0$

$u_g(t) = C_1 + C_2 e^{-5t}$

$u'' + 5u' = 3e^t + 5$

$u_p(t) = Ae^t + Bt$

$u_p'(t) = Ae^t + B$

$u_p''(t) = Ae^t$

$Ae^t + 5Ae^t + 5B = 3e^t + 5$

$6A = 3 \quad B = 1$

$A = \frac{1}{2}$

$\left[u(t) = C_1 + C_2 e^{-5t} + \frac{1}{2}e^t + t \right]$

$$\begin{cases} (D+1)u - (D-1)v = e^t \\ (D-1)u - (2D+1)v = 5 \end{cases}$$

$$-\begin{cases} (D-1)(D+1)u - (D-1)^2 v = e^t(D-1) \\ (D+1)(D-1)u - (D+1)(2D+1)v = (D+1)5 \end{cases}$$

$[-(D-1)^2 + (D+1)(2D+1)]v = (D-1)e^t - (D+1)5$

$[-D^2 + 2D - 1 + 2D^2 + 3D + 1]v = e^t - e^t - 5$

$[D^2 + 5D]v = 5$

$B_1 + B_2 e^{-5t} = v_p(t)$

$v_p = At \qquad A = 1$

$\left[v(t) = B_1 + B_2 e^{-5t} + t \right]$

$u(t) = C_1 + C_2 e^{-5t} + \frac{1}{2}e^t + t$

$v(t) = B_1 + B_2 e^{-5t} + t$

② a $y''' + 2y'' - 11y' - 12y = 0$, $\{e^{3x}, e^{-x}, e^{-4x}\}$

$W[e^{3x}, e^{-x}, e^{-4x}]$

$$\begin{vmatrix} e^{3x} & e^{-x} & e^{-4x} \\ 3e^{3x} & -e^{-x} & -4e^{-4x} \\ 9e^{3x} & e^{-x} & 16e^{-4x} \end{vmatrix}$$

$e^{3x}[-16e^{-5x} + 4e^{-5x}] - e^{-x}[48e^{-x} + 36e^{-x}] + e^{-4x}[3e^{2x} + 9e^{2x}]$

$= e^{3x}[-12e^{-5x}] - e^{-x}[84e^{-x}] + e^{-4x}[12e^{2x}]$

$\cancel{-12e^{-2x}} - 84e^{-2x} + \cancel{12e^{-2x}} = -84e^{-2x}$

try one, will work

$y(x) = C_1 e^{3x} + C_2 e^{-x} + C_3 e^{-4x}$

b) Same Shit

③ (a) $(D-1)^2(D+3)(D^2+2D+5)^2[y] = 0$

$r^2 + 2r + 5 = 0$

$\frac{-2 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = \frac{-2 \pm \sqrt{4}\, i}{2} = \frac{-2 \pm 4i}{2} = 1 \pm 2i$

$y(t) = C_1 e \ldots$

③ b) $(D+4)(D-3)(D+2)^3(D^2+4D+5)^2 D^5[y] =$

$r^2 + 4r + 5 = 0$

$\frac{-4 \pm \sqrt{16 - 4 \cdot 1 \cdot 5}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$

$y(t) = C_1 e^{-4t} + C_2 e^{3t} + t^2 C_3 e^{-2t} + te^{t}(C_4 \cos t + C_5 \sin t) + C_6 + C_7 t + C_8 t^2 + C_9 t^3 + C_{10} t^4 + C_{11} t^5 + C_{12} t^6$

$D^5 = \text{Poly } n - 1 = P(4)$

$\int_0^\infty e^{-st} f(t)\, dt$

④ a) $u'' - 5u' + 6u = \cos(2x) + 1$

$(D^2 - 5D + 6)[u] = \cos 2x + 1$

$(D^2 - 5D + 6)(D^2 + 4)D[u] = 0$

$r^3 + 4r = 0$

$r(r^2 + 4) = 0$

$C_1 + C_2 \cos 2t + C_3 \sin 2t = u_p$

$u_p' = -2C_2 \sin 2t + 2C_3 \cos 2t$

$u_p'' = -4C_2 \cos 2t - 4C_3 \sin 2t$

$-4C_2 \cos 2t - 4C_3 \sin 2t$

$10C_2 \sin 2t - 10C_3 \cos 2t$

$6C_2 \cos 2t + 6C_3 \sin 2t + 6C_1$

$(2C_2 - 10C_3)\cos(2t)$

$+ (10C_2 + 2C_3)\sin(2t) + 6C_1$

$$\begin{bmatrix} 2 & -10 \\ 10 & 2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad 6C_1 = 1,\ C_1 = 1/6$$

$u_p = \frac{1}{52}\cos 2t - \frac{5}{52}\sin 2t + \frac{1}{6}$

b) $y'' + 2y' + 2y = e^{-x}\cos x + x^2$

Anihilation time! Solve Zero First

$((D+1)^2 + 1)D^3$

$((r+1)^2 + 1)r^3 = 0$

$(r^2 + 2r + 2)r^3 = 0$

$r = 0_3$,

$r^2 + 2r + 1 = -1$

$(r+1)^2 = -1$

$r + 1 = \pm i$

$r = -1 \pm i$

$A_0 + A_1 x + A_2 x^2 + xe^{-x}(B_1 \cos x + B_2 \sin x)$

⑤ $y''' + 2y'' - 9y' - 18y = -18x^2 - 18x + 22$

$y(0) = -2$, $y'(0) = -8$, $y''(0) = -12$

$r^3 + 2r^2 - 9r - 18 = 0$

3	1	2	−9	−18
	↓	3	−3	−18
	1	−1	−6	0

$(r^2 - r - 6)(r - 3) = 0$

$(r-3)(r-2)(r-3) = 0$

$(r-3)^2(r-2) = 0$

$y_p = (C_1 + C_2 x)e^{3x} + C_3 e^{2x}$

$y_p' = (C_2 + 3C_1 + 3C_2 x)e^{3x} + 2C_3 e^{2x}$

$y_p' = (3C_1 + (1 + 3x)C_2)e^{3x} + 2C_3 e^{2x}$

$y'' = (3C_2 + 9C_1 + 9C_2 x)e^{3x} + 4C_3 e^{2x}$

$y''' = (9C_2 + 27C_1 + 27C_2 x)e^{3x} + 8C_3 e^{2x}$

$[\cancel{27}C_1 + \cancel{9}C_2 + \cancel{27}C_2 x]e^{3x} + 8C_3 e^{2x}$

$[\cancel{18}C_1 + 6C_2 + \cancel{18}C_2 x]e^{3x} + 8C_3 e^{2x}$

$[-\cancel{27}C_1 - \cancel{9}C_2 - \cancel{27}C_2 x]e^{3x} - 18C_3 e^{2x}$

$[-\cancel{18}C_1 - \cancel{18}C_2 x]e^{3x} - 18C_3 e^{2x}$

$= 6C_2 e^{3x} - 20C_3 e^{2x}$? why? :(

$C_1 + C_3 = -2$

$3C_1 + C_2 + 2C_3 = -8$

$9C_1 + 3C_2 + 4C_3 = -12$

$$\begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \\ 9 & 3 & 4 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -8 \\ -12 \end{bmatrix} \rightarrow \begin{matrix} C_1 = 4 \\ C_2 = -8 \\ C_3 = -6 \end{matrix}$$

$y_g(x) = (4 - 8x)e^{3x} - 6e^{2x}$

$y_p = A_0 + A_1 x + A_2 x^2$

$y_p' = A_1 + 2A_2 x$

$y_p'' = 2A_2$

$y_p''' = 0$

$4A_2$

$-9A_1 - 18A_2 x$

$\underbrace{-18A_0}_{22} \underbrace{- 18A_1 x}_{-18x} \underbrace{- 18A_2 x^2}_{-18}$

$$\left[\begin{array}{ccc|c} -18 & 0 & 0 & -18 \\ -18 & -9 & 0 & -18 \\ 4 & -18 & -18 & 22 \end{array}\right] = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$y(x) = (4 - 8x)e^{3x} - 6e^{2x} + x^2 + 1$

⑥ a) $e^{2t}\cos 3t$

$I = \int_0^\infty e^{-st} e^{2t} \cos 3t\, dt = \int_0^\infty e^{(2-s)t} \cos 3t\, dt$

$dv = e^{(2-s)t} \qquad u = \cos 3t$

$v = \frac{1}{2-s} e^{(2-s)t} \qquad du = -3 \sin 3t$

$I = \left[\frac{\cos 3t}{2-s} e^{(2-s)t} \right]_0^\infty - \frac{3}{s-2} \int_0^\infty e^{(2-s)t} \sin 3t\, dt$

$\lim_{t\to\infty} \frac{\cos 3t}{2-s} e^{(2-s)t} = \frac{1}{2-s} \lim_{t\to\infty} \cos 3t \cdot e^{(2-s)t} = 0$

$s > 2$

$I = -\frac{3}{s-2} \int_0^\infty e^{(2-s)t} \sin 3t\, dt$

$u = \sin 3t \qquad dv = e^{(2-s)t}$

$du = 3\cos 3t \qquad v = \frac{1}{2-s} e^{(2-s)t}$

$I = -\frac{3}{s-2} - \frac{3}{s-2} \left[\left(\frac{\sin 3t}{2-s} e^{(2-s)t} \right) \Big|_0^\infty - \frac{3}{2-s} I \right]$

$I = -\frac{3}{s-2} - \frac{3}{s-2} \left(\frac{-3}{2-s} I \right)$

$I = -\frac{3}{s-2} + \frac{3}{2-s} \left(\frac{-3}{2-s} I \right)$

$I = -\frac{3}{s-2} + \frac{9}{(2-s)^2} I$

$I - \frac{9}{(2-s)^2} I = \frac{3}{2-s}$

$I = \frac{3}{2-s} \left(1 - \frac{9}{(2-s)^2} \right)^{-1}$

$I = \frac{3}{2-s} \left(\frac{(2-s)^2 - 9}{(2-s)^2} \right)^{-1}$

$I = \frac{3}{\cancel{2-s}} \left(\frac{(2-s)^{\cancel{2}}}{(2-s)^2 - 9} \right)$

$I = \frac{3(2-s)}{(2-s)^2 - 9}$

⑥ b) te^{3t}

$\int_0^\infty e^{-st} te^{3t}\, dt = \int_0^\infty e^{(3-s)t} t\, dt$

$u = t \qquad dv = e^{(3-s)t}$

$du = 1 \qquad v = \frac{1}{3-s} e^{(3-s)t}$

$I = \left[\frac{te^{(3-s)t}}{(3-s)} \Big|_0^\infty \right] - \frac{1}{3-s} \int_0^\infty e^{(3-s)t}\, dt$

for $t = 0 \to 0$ $\quad \lim_{t\to\infty} f(t) = 0$ for $s > 3$ $\quad \checkmark \frac{e^{3t}}{e^{st}}$

$I = -\frac{1}{3-s} \left(\underbrace{\frac{e^{(3-s)t}}{3-s} \Big|_0^\infty}_{0} \right) = 0$?

⑦ a) $\mathcal{L}\{6e^{-3t} - t^2 + 2t - 8\}$

$= \frac{6}{s+3} - \frac{2}{s^3} + \frac{2}{s^2} - \frac{8}{s}$

b) $\mathcal{L}\{t^4 e^{5t} - e^t \cos(t\sqrt{7})\}$

$= \frac{24}{(s-5)^5} - \frac{s-1}{(s-1)^2 + 7}$

⑧ $F(s) = \frac{6s^2 - 13s + 2}{s(s-1)(s-6)}$

$6s^2 - 13s + 2 = A(s-1)(s-6) + B(s-6)s + C(s-1)s$

$A(s^2 - 7s + 6) + B(s^2 - 6s) + C(s^2 - s)$

$As^2 + Bs^2 + Cs^2 - 7sA - 6sB - Cs + A6$

$$\begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ -7 & -6 & -1 & -13 \\ 6 & 0 & 0 & 2 \end{array}\right] \end{array} = \begin{bmatrix} 1/3 \\ 1 \\ 14/3 \end{bmatrix}$$

$F(s) = \frac{1}{3}\frac{1}{s} + \frac{1}{s-1} + \frac{1}{3}\frac{14}{s-6}$

$\left[\frac{1}{3}(1) + e^t + e^{6t} t \right] = \mathcal{L}^{-1}[f(s)]$

b) $7s^2 - 41s + 84 = A(s^2 - 4s + 13) + (Bs + C)(s - 1)$

$s^2 A - 4sA + 13A$

$s^2 B - sB + sC - C$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ -4 & -1 & 1 & -41 \\ 13 & 0 & -1 & 84 \end{array}\right] = \begin{bmatrix} 5 \\ 2 \\ -19 \end{bmatrix}$$

$\frac{7s^2 - 41s + 84}{(s-1)(s^2 - 4s + 13)} = \frac{A}{s-1} + \frac{Bs + C}{s^2 - 4s + 13}$

$= \frac{5}{s-1} + \frac{2s - 19}{s^2 - 4s + 13}$

$s^2 - 4s + 13 = 0$

$s^2 - 4s + 4 = -9$

$(s-2)^2 + 9 = 0$

$= \frac{5}{s-1} + \frac{2s - 19}{(s-2)^2 + 3^2} =$

$= \frac{5}{s-1} + 2\frac{s-2}{(s-2)^2 + 3^2} - \frac{15}{(s-2)^2 + 3}$

$= \frac{5}{s-1} + 2\frac{s-2}{(s-2)^2 + 3^2} - 5\frac{3}{(s-2)^2 + 3}$

$\mathcal{L}^{-1}[F(s)] = 5e^t + e^{2t}\cos 3t + e^{2t}\sin 3t$

⑨ $y'' + 6y' + 9y = 0 \quad y(0) = -1 \quad y'(0) = 6$

$\mathcal{L}\{y'' + 6y' + 9y\} = \mathcal{L}\{0\}$

$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$

$= s^2\mathcal{L}\{y\} + s - 6$

$\mathcal{L}\{6y'\} = 6s\mathcal{L}\{y\} - 6y(0)$

$= 6s\mathcal{L}\{y\} + 6$

$(s^2 - 6s + 9)\mathcal{L}\{y\} + s = 0$

$\mathcal{L}\{y\} = -\frac{s}{s^2 - 6s + 9}$

$= \frac{-s}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$

$\frac{A(s-3) + B}{(s-3)^2} = \frac{-s}{s-3}$

$As - 3A + B = -s$

$As = -1 \qquad +3 + B = 0$

$A = -1 \qquad B = -3$

$\mathcal{L}\{y\} = \frac{-3}{(s-3)^2} - \frac{1}{(s-3)}$

$-3e^{3t}t - e^{3t} = -(3t+1)e^{3t}$

b) $z'' + 5z' - 6z = 21e^{t-1}$

$z'' + 5z - 6z = \frac{21}{e} e^t$

$\mathcal{L}\{z'' + 5z' - 6z\} = \mathcal{L}\{\frac{21}{e} e^t\}$

$\mathcal{L}\{z''\} + 5\mathcal{L}\{z'\} - 6\mathcal{L}\{z\} = \frac{21}{e}\mathcal{L}\{e^t\}$

$s^2\mathcal{L}\{z\} - sy(0) - y'(0)$

$-5s\mathcal{L}\{z\} - 5y'(0)$

$-6\mathcal{L}\{z\} = \frac{21}{e}\frac{1}{s-1}$

$(s^2 - 5s - 6)\mathcal{L}\{z\} - s(-1) - (6) - 5(6)$

$= (s^2 - 5s - 6)\mathcal{L}\{z\} + s - 36 = \frac{21}{e}\frac{1}{s-1}$

$= (s^2 - 5s - 6)\mathcal{L}\{z\} = \frac{21}{e}\frac{1}{s-1} + 36 - s$

$\mathcal{L}\{z\} = \frac{21}{e}\frac{1}{(s^2 - 5s - 6)(s-1)} + \frac{36}{(s^2 - 5s - 6)} - \frac{s}{(s^2 - 5s - 6)}$