1 - Background

- 1.1 Introduction
  - Differential Equation: An equation that contains a derivative.
    - Types of differential equations.
      - Ordinary Differential Equation: It has only one ordinary derivative with a single independent variable
        - Ordinary Derivative: A derivative with respect to a single variable. e.g. $\frac{dy}{dx}$
      - Partial Differential Equation: An equation involving partial derivatives and more than one independent variable.
        - Partial Derivative: like calc III derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ etc.
      - Linear Differential Equation: the derivatives come as combinations of their powers.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_0(x)y = F(x)$$

See how the x coefficients must be independent in order to be linear.

- Order: refers to the highest derivative value

- Classifying DE
  1 - Three types
  - Linearly Ordinary DE
  - Non Linearly, ordinary DE
  - Partial DE

  2 - Linear / non linear Ordinary
  Linear if
  - The orders of the derivatives go down can't miss a number, no repetitions
    - Tip: Distribute parentesis
  - No dependent variables as coefficients
  - Look for the notation, no $\frac{\partial z}{\partial y}$ only $\frac{dy}{dx}$

  Else non linear ordinary or PDE.

  3 - if there have multiple derivatives with respect to different variables then it is a Partial Derivative
  else → ODE Non linear

- 2.4 Exact Equations

Shows in the form of

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

There exist a function $F(x,y)$ in a region R such that

$$\frac{\partial F}{\partial x}(x,y) = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x,y) = N(x,y)$$

- To test for exactness, this needs to be true

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)$$

- Solving Exact equations

I Verify

Exam 1

- **Directional Fields**
  They are a graphical represeatation of differential equatios.

- **Solving Directional field problems**
  1- Drop the point
  2- Connect the slope lines
  3- Moving $x \to \infty$ left to right
  Moving $x \to -\infty$ right to left
  4- Draw how it behaves.
  5- Draw a conclusion.

- **Euler's Method**
  To find the value of a point in an equation just from the differential equation alone.
  A method for approximation.

Euler's Method

① state the variables

x_0, y_0, h

② Use a table and calculator

$$x_n \quad y_n \quad y_{n+1} = y_n + h f(x_n, y_n)$$

③ Solve the following initial value problems.

- **Separable Differential Equations**
  If the given differential equation can be expressed as

$$\frac{dy}{dx} = g(x)\, p(y)$$

then it is separable

- **Solving separable Differential Equations**
  1 - express as: to solve

$$h(y)\, dy = g(x)\, dx$$

  2 - Integrate

$$\int h(y)\, dy = \int g(x)\, dx$$

  3 - $H(y) = G(x) + C$

- **Linear Differential Equation**
  a first order linear differential equation is of the form

$$a_1(x) \frac{dy}{dx} + a_0(x) y = b(x)$$

- **Method for solvig First order linear Diff eq...**
  1- write in standard form

$$\frac{dy}{dx} + P(x) y = Q(x)$$

  No values multiplying $\frac{dy}{dx}$

  2 - Identify the $P(x)$
  e.g.

$$\frac{dy}{dx} + 4y = e^{-x}$$

  here $P(x) =$ is 4

  -Tip: keep the constant for $P(x)$ leave the y.

  3 - Integrate $P(x)$ and exponentiate

$$\mu(x) = e^{\int P(x)\, dx} \qquad \text{P.S. No } C$$

  4 - Multiply through by $\mu(x)$

$$\mu(x) \frac{dy}{dx} + P(x)\mu(x) y = \mu(x) Q(x)$$

  This is the backwards to the product rule

$$\frac{d}{dx}[\mu(x) y] = \mu(x)\, Q(x)$$

  5 - Integrate

$$\mu(x) y = \int \mu(x) Q(x)\, dx$$

  Note If we have an initial value here after the integration keep the C and solve for y.

- **Exact equations**
  The slope of a level curve is an exact equation I guess...

  It has the form of

$$M(x,y)\, dx + N(x,y)\, dy = 0$$

- **Test for exactness**
  1

$$\frac{\partial}{\partial y} M(x,y) = \frac{\partial}{\partial x} N(x,y)$$

  If true exact, else not exact. Do not proceed with integration.

- **Solving Exact Equations**
  1 - Pick $M(x,y)$ and integrate with respect to x

$$\int M(x,y)\, dx + g(y) = F(x,y)$$

  2 - Take the partial derivative with respect to y and set to $N(x,y)$

$$\frac{\partial}{\partial y}[M(x,y)] + g'(y) = N(x,y)$$

  3 - Cancell out terms and solve for $g'(y)$

  4 - Integrate $g'(y)$ to obtain $g(y)$ and sub into original equation

  5 Final answer looks like

$$\int M(x,y)\, dx + g(y) = C$$

- **Compartmental Analysis**
  The amount of something in a compartment $x(t)$ and the amount of time t.

$$\frac{dx}{dt} = \text{Input} - \text{output}$$

- **Solving Mixing Problems**
  1- Find what we are looking for
  - The amount of something in a compartment

  2- $\frac{dx}{dt}$ = Input rate — output rate

  2.1
  input rate = Consentration × Input rate

  2.2 Output rate = $\frac{\text{Substance}}{\text{total compartment}}$ × output rate.
  Solution %
  Substance = x
  Total compartment =
  Tank initial volume + (input rate - OutputRate)(t)

  3- Solve using First order Diff steps

  4- Initial value to find C

  5- Answer t time questions.

- **Improved Euler's Method**
  Same as euler's method, just better!

- **Using Improved Method**
  1 - Use

$$x_{n+1} = h + x_n$$

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_n + h, y_n + h f(x_n + h))]$$

  2 - Make a table with the following

x_n	y_n	$f(x_n, y_n)$	$y_n + hf(x_n+h)$	$f(x_n+h,\ y_n+hf(x_n+h))$
x_{n+1}	y_{n+1}			
x_{n+2}				
⋮				

Keep work clean

Exam #2

- Homogenious equations
  $y'' + ay' + cy = 0$

- Solving homogenios equations

1- write the characteristic equation and solve

2- if the rooths are real and different then

$c_1 e^{at} + c_2 e^{bt} = y(t)$

3- Similar but real

$c_1 e^{at} + t c_2 e^{at}$

3- Imaginary divverent

$\alpha \pm \beta i = r$

$e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$

" imaginary Same

$\pm \beta i = r$

$c_1 \cos(\beta t) + c_2 \sin(\beta t)$

- Particular solutions

$y'' + by' + cy = f(x)$

1- Find the homogenius equation

2- guess if trig

$A \sin \beta t + B \cos \beta t$

3- Exponential

$A e^{\beta t}$

4- Polynomial

$(A_0 + A_1 t + \cdots + A_n t^n)$

5 multiply by a factor of t^n if a particular solution is the same as the general solution

- Variation of parameters

1 Solve the characteristic equation this yields y_1 and y_2

2 derivate y_1 and y_2

3 Find

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W$$

4 do

$$v_1 \int \frac{-y_2 \cdot f}{W}, \quad v_2 = \int \frac{y_1 f}{W}$$

5 your answer

$y = c_1 y_1 + c_2 y_2 + v_1 y_1 + v_2 y_2$

- Cauchy Euler

1- your guess $y = t^r$ $y' = r t^{r-1}$, $y'' = r(r-1)t^{r-2}$

2 plug in

2 solve equation $c_1 t^{r_1} + c_2 t^{r_2}$

$r = a, b$

If rooths are complex

$\alpha \pm \beta i$

$y_1 = t^\alpha \cos \beta \ln t$ $\quad y_2 = t^\alpha \sin \beta \ln t$

$y = t^\alpha (c_1 \cos \beta \ln t + c_2 \sin \beta \ln t)$

if double rooth

$y_1 = t^r$ $\quad y_2 = t^r \ln t$

$y = c_1 t^r + c_2 t^r \ln t$

Extra on integrals

$\int \tan x \, dx = \ln|\sec x| + C$

$\int \cot x \, dx = \ln|\sin x| + C$

$\int \sec x \, dx = \ln|\sec x + \tan x| + C$

$\int \csc x \, dx = -\ln|\csc x + \cot x| + C$

$\sin^2 x = \frac{2 - \cos 2x}{2}$

$\cos^2 x = \frac{2 + \cos 2x}{2}$