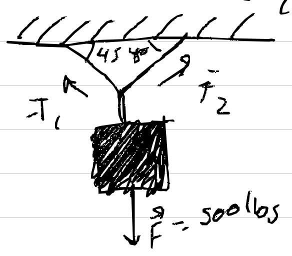
- Class notes Calculus III 18-2-25

e.g. A 500 lbs thing is suspended by two chains that make a 45° angle with the ceiling. What is the magnitude of the force that each chain must support?

Already seen.



$F = \langle 0, -500 \rangle$

$\vec{F}_1 + \vec{F}_2 + \vec{F} = 0$

$\vec{F}_1 = \langle -|F_1| \cos(\frac{\pi}{4}), |F_1| \sin(\frac{\pi}{4}) \rangle$

$\vec{F}_2 = \langle |F_2| \cos(\frac{\pi}{4}), |F_2| \sin(\frac{\pi}{4}) \rangle$

$-\frac{\sqrt{2}}{2}|F_1| + \frac{\sqrt{2}}{2}|F_2| = 0$

$|F_1| = |F_2|$

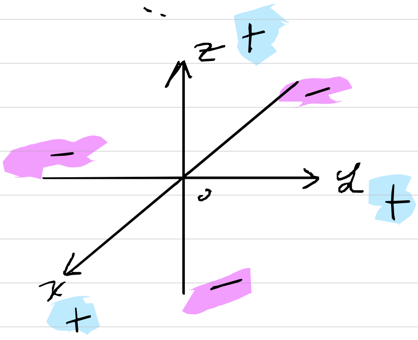
$|F_2| \frac{\sqrt{2}}{2} + |F_1| \frac{\sqrt{2}}{\sqrt{2}} = 500$ lbs

$2|F_1|(\frac{\sqrt{2}}{2}) = 500$ lbs

$F_1 = \frac{500}{\sqrt{2}}$ lbs $= \sqrt{2}\,500$ lbs

13.2 Vectors in Space

- 13.2 Vectors in 3 dimensions
  - Vectors in space.
- Ordered triplets $(x, y, z) \rightarrow \mathbb{R}^3$
- Right hand rule for Positive
  - Tradition
- Octanes A 1/8 of the 3D plan
  e.g. $(+,+,+)$, $(-,+,+)$, $\cdots$ $(-,-,-)$
  - octans have no numbered order except from the 1th: $(+,+,+)$



- Magnitude of a vector

$\vec{v} = \langle a, b, c \rangle$ $\quad \sqrt{a^2+b^2} = p$ $\quad \sqrt{p^2+c^2} = |\vec{v}|$

$|\vec{v}| = \sqrt{a^2+b^2+c^2}$ — Euklidean Rule?

- Distance between two points
  $p(x_1, y_1, z_1)$, $q(x_2, y_2, z_2)$
  $a = x_2 - x_1$
  $b = y_2 - y_1$ $\quad \sqrt{a^2+b^2+c^2} = \vec{pq}$
  $c = z_2 - z_1$
- Sphere formula
  A sphere with radius r and centered at (x_0, y_0, z_0) is
  $r^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2$
- e.g. Find the center and radius of the sphere
  $x^2 + y^2 + z^2 - 2x + 6y - 8z = -1$

$(x^2 - 2x + 1 - 1) + (y^2 + 6y + 9 - 9) + (z^2 - 8z + 16 - 16) = -1$

$(x-1)^2 - 1 + (y+3)^2 - 9 + (z-4)^2 - 16 = -1$

$(x-1)^2 + (y+3)^2 + (z-4)^2 = 25$

$r = 5$ and it's centered at $(1, -3, 4)$

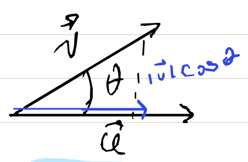
- steps
  1 - Complete the squares
  2 - Clean the constants
  3 - Write the center and r
- 3D vectors
- If $|\vec{v}| = 1$, then $\vec{v}$ is a unit vector
  - Unit vectors help us with direction.
  - Since the only information is the angle.
- Discribing vectors
  $\vec{v} = |\vec{v}|\hat{u}$ the $\hat{u}$ is like the compass telling where to go.
- Unit vectors
  $\hat{i} = \langle 1, 0, 0 \rangle$, $\hat{j} = \langle 0, 1, 0 \rangle$, $\hat{k} \langle 0, 0, 1 \rangle$

13.3 Vector Multiplication

- 13.3 Vector Multiplication
- Dot Product
  $\vec{v} \cdot \vec{u} = |\vec{u}||\vec{v}| \cdot \cos\theta$
  where θ is the angle between $\vec{v}$ and $\vec{u}$.
  and $0 \le \theta \le \pi$

AKA Work
$|d||F| \cdot \cos\theta$

- Definition
  Two vectors are orthogonal if their dot product is zero
  $\vec{v} \cdot \vec{u} = 0$



$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$

From Law of cosines
- Short cut to not compute θ

- e.g. $\vec{u} = \langle u_1, u_2, u_3 \rangle$
  $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$\vec{u} \cdot \vec{v} = \frac{1}{2}(|\vec{u}|^2 + |\vec{u}|^2 - |u - v|^2)$

- Expanding we obtain (some how)
  $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

Caculus III Lecture Notes

20-2-25

- Review of previus class
- How to do a dot product

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = c$

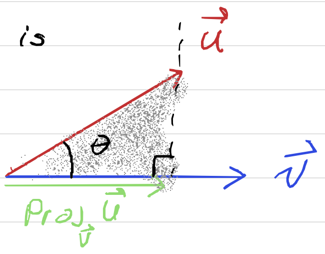
- How to find the angle of two vectors

$$\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \cos\theta$$

- Properties of the dot product
  1 - $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
  2 - $c(\vec{u} \cdot \vec{v}) = c\vec{u} \cdot c\vec{v}$
  3 - $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

- Orthogonal Projection

The scalar projection of $\vec{u}$ onto $\vec{v}$ is the magnitude of $proj_{\vec{v}}\vec{u}$.



$$|\vec{u}|\cos\theta = scal_{\vec{v}}\vec{u}$$

- Since θ is hard to find
It is used a unit vector in the direction of $\vec{v}$ and that is $\frac{\vec{v}}{|\vec{v}|}$

The projection of $\vec{u}$ onto $\vec{v}$ is guiven by

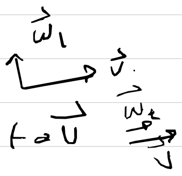
$$proj_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\vec{v}$$

And the scalar is

$$scal_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}$$

- $\vec{v} \cdot \vec{v} = |\vec{v}|^2$

- e.g. $\vec{u} = 3\hat{i} - 2\hat{j} + 5\hat{k}$ write $\vec{u}$ as $\vec{\omega}_1 + \vec{\omega}_2$
$\vec{v} = \hat{i} + \hat{j} + 2\hat{k}$ where $\vec{\omega}_1$ paralel to $\vec{v}$
and $\vec{\omega}_2$ is orthogonal to $\vec{v}$



$\vec{u} \cdot \vec{v} = 3 - 2 + 10 = 11$
$|\vec{v}|^2 = 6$

$proj_{\vec{v}}\vec{u} = \frac{11}{6}(\hat{i} + \hat{j} + 2\hat{k})$

$(11/6, \ 11/6, \ 11/3) = \omega_1$

$\omega_2 = \vec{u} - \vec{\omega}_1 = (7/6, -23/6, 4/3)$

$$\boxed{\begin{aligned} \vec{\omega}_1 &= (11/6, \ 11/6, \ 11/3) \\ \vec{\omega}_2 &= (7/6, \ -23/6, \ 4/3) \end{aligned}}$$

- e.g. A constant force $\vec{F} = \langle 4, 3, 2 \rangle$ moves an object from $(0,0,0)$ to $(8,6,0)$, calculate work done

$W = \vec{F} \cdot \vec{w}$

$$\boxed{32 + 18 + 0 = 50 \text{ J}}$$

13.4 Cross product.

- Torque formula

$$\vec{T} = |\vec{r}||\vec{F}| \sin\theta = |\vec{r} \times \vec{F}|$$

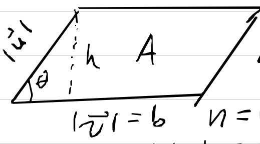
- Cross product formula

$\vec{u}, \vec{v} \in \mathbb{R}^3$

$$\vec{u} \times \vec{v} = |\vec{u}||\vec{v}| \sin\theta$$

Where θ is the angle between $\vec{u}$ and $\vec{v}$. The direction is guiven by the "right hand" rule.
- Curl fingers towards the vector being crossed on the thumb points towards the direction.

- Theorem
  1 - $\vec{u} \times \vec{v} = 0$ then $\vec{u}$ and $\vec{v}$ are paralel.
  2 - $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
  3 - The area of a puralelogram formed by $\vec{u}$ and $\vec{v}$ is the cross product.



$A = b * h$
$|\vec{v}| = b$ $h = |\vec{u}| \sin\theta$
$|\vec{u}||\vec{v}| \sin\theta = A = \vec{u} \times \vec{v}$

- Properties of the cross product.
  1 - $(a\vec{u}) \times (b\vec{v}) = ab(\vec{u} \times \vec{v})$
  2 - $\vec{u} \times (\vec{v} + \vec{\omega}) = \vec{u} \times \vec{v} + \vec{v}\ \vec{\omega}$
  3 - $(\vec{u} + \vec{v}) \times \vec{\omega} = \vec{u} \times \vec{\omega} + \vec{v}\ \vec{\omega}$

- Other properlies

$\hat{i} \times \hat{j} = \hat{k}$ $\hat{j} \times \hat{i} = -\hat{k}$
$\hat{j} \times \hat{k} = \hat{i}$ $\hat{k} \times \hat{j} = -\hat{i}$
$\hat{k} \times \hat{i} = \hat{j}$ $\hat{i} \times \hat{k} = -\hat{j}$

RLN
Notes in yellow.

Lecture Notes 25-2-25

- Cross product example

$\vec{u} = 3\hat{i} - 2\hat{j} + 4\hat{k}$
$\vec{v} = \hat{i} + 3\hat{j} - 3\hat{k}$

$\vec{u} \times \vec{v}$

$\cancel{3\hat{i} \times \hat{i}} \to 0 + 3\hat{i} \times 3\hat{j} + 3\hat{i} \times -3\hat{k}$
$+ -2\hat{j} \times \hat{i} \; \cancel{-2\hat{j} \times 3\hat{j}} \to 0 \; -2\hat{j} \times -3\hat{k}$
$+ 4\hat{k} \times \hat{i} + 4\hat{k} \times 3\hat{j} \; \cancel{+4\hat{k} \times -3\hat{k}} \to 0$

Using previously discussed properties the unit vectors have -1, or 1

$\hookrightarrow = -6\hat{i} + 13\hat{j} + 11\hat{k}$

- Cross product from a determinant

$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix}$ "works because"

– Order does matters

- e.g. Given 3 points find the area of the triangle

$P \langle 1, 2, 3 \rangle$
$Q \langle 3, 2, 1 \rangle$
$R \langle 7, 8, 9 \rangle$

Basically choose $\vec{PQ}, \vec{PR}$ ($\vec{u}$, $\vec{v}$) do $\vec{u} \times \vec{v}$

...

$|4\hat{i} + 2\hat{j} + 2\hat{k}| = 2\sqrt{6}$ This is the area of the paralellogram Just divide by 2

$\boxed{A = \sqrt{6}}$

- e.g. – Find a vector orthogonal to both $\vec{u} = 2\hat{i} - 4\hat{k}$ and $\vec{v} = \hat{i} + \hat{j} + \hat{k}$

– Take their cross product

$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -4 \\ 1 & 1 & 1 \end{vmatrix} \dots =$

- **13.5 Lines and planes in space.**
- Vector equation of a line passing through the point $P_0(x_0, y_0, z_0)$ in the direction of $\vec{v} = \langle a, b, c \rangle$ is

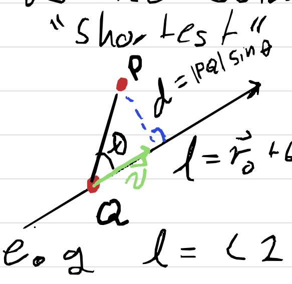
$\vec{r}(t) = P_0 + t\vec{v_0} \Leftrightarrow \langle x_0 + at, y_0 + bt, z + bt \rangle$

- e.g find the equation of a line passing trough $P \langle 1, 5, -8 \rangle$, $Q(7, 2, 4)$

$P_0 = (1, 5, -8)$
$\vec{PQ} = V_0 = (6, -3, 12) \to \vec{r}(t) = P_0 \, t \vec{v_0}$

1- find $\vec{PQ}$ or $\vec{QP}$ order does not matter.
2- write $\vec{r}(t) = P_0 + t\vec{v}$

- distance Between a poin and a line
"shortest"



$\frac{|\vec{v}||\vec{PQ}| \sin\theta}{|\vec{v}|} = \frac{|\vec{v} \times \vec{PQ}|}{|\vec{v}|} = d$

- e.g $l = \langle 2, -3, 8 \rangle + t \langle 1, 3, 2 \rangle$, $Q(1, 3, 4)$

$\vec{PQ} = \langle -1, 6, -4 \rangle$

$\vec{v} \times \vec{PQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 2 \\ -1 & 6 & -4 \end{vmatrix} = \dots = -24\hat{i} + 2\hat{j} + 9\hat{k} \overset{mag}{\Rightarrow} \sqrt{14}$

$|\vec{v}| = \sqrt{661}$ $\quad d = \frac{\sqrt{661}}{\sqrt{14}}$

– Using calculus

$d = \sqrt{(1 - (2+t))^2 + (3 - (-3+3t))^2 + (4 - (8+2t))^2}$

- e.g Do these lines intersect?

$l_1 = \langle 2, 0, 1 \rangle + t \langle 3, 3, -1 \rangle$
$l_2 = \langle 4, -3, 0 \rangle + s \langle 2, 3, -2 \rangle$

- 1 express as equations

$2 + 3t = 4 + 2s$
$0 + 3t = -3 + 3s$
$1 - t = 0 + -2s$

$\Rightarrow \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} + v \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$

② Use equations to find t and s
③ use the third equation to verify if they intersect.

- **Def** Given a point $P(x_0, y_0, z_0)$ and a non zero vector $\vec{n} = \langle a, b, c \rangle$ the set of points $Q(x, y, z)$ for which $\vec{PQ}$ is orthogonal is called a plane.

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

e.g. $P(1, 3, 7)$, $\vec{n} = \langle 2, 4, 2 \rangle$

$2(x - 1) + 4(y - 3) + 7(z - 2) = 0$

e.g. Find an equation of a plane containing $P(2, 0, 5)$, $Q(-3, 1, 5)$, $R(1, 1, 1)$.

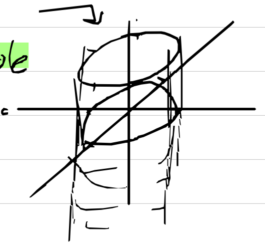
① choose a point (Any one works)
② find the normal vector (find the orthogonal vector)
- Use cross product.
- Create two vectors
$\vec{PQ} = \langle -5, 1, 0 \rangle$
$\vec{PR} = \langle -1, 1, -4 \rangle$

③ Cross product of $\vec{PR} \times \vec{RQ} = 4\hat{i} + 20\hat{j} + 4\hat{k}$
$\vec{n} = 4\hat{i} + 20\hat{j} + 4\hat{k}$
④ write the equation

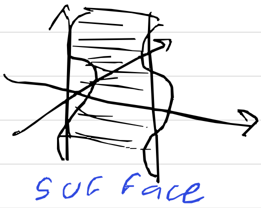
$4(x - 1) + 20(y - 3) + 4(z - 7) = 0$

Calculus III 27-2-25

- 13.6 Cylinders and quadratic surfaces
- e.g. $x^2+y^2=16$



- An equation that's missing a variable in 3D space is called a cylinder.
- e.g. $x - \sin z = 0$



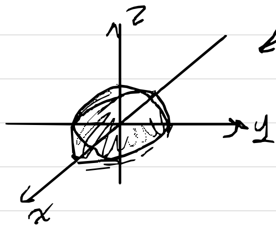
- Quadric surface

A quadric surface is the graph of an equation of the form:

$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+J=0$$

- e.g. An ellipsoid $\frac{x^2}{9}+\frac{y^2}{25}+\frac{z^2}{16}=1$
  - using traces
    ↳ If you take a plain and intersect it you obtain a traise.

Some sort of fucked up sphere

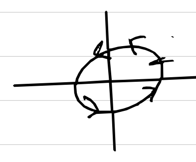


Lecture Notes 4-3-25

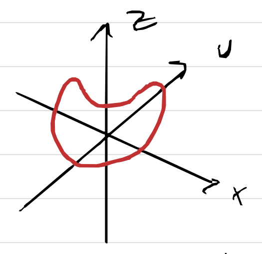
- 14 Vector valued functions

e.g. $\vec{r}(t) = \langle 2+3t, 4-t, 1+t \rangle$

- Already seen, just as parametric functions.
- e.g. $f(x) = \langle \cos t, \sin t \rangle$
  - It has an orientation



- $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 0.4\sin(2t)\hat{k}$



- Definition of a limit for a vector valued functions.

A vector valued function $\vec{r}$ approaches the limit $\vec{L}$ as t approaches a

$$\lim_{t\to a} \vec{r}(t) = \vec{L}$$

- Provided that $\lim_{t\to a} |\vec{r}(t) - L| = 0$

- Theorem if $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

and $\lim_{t\to a} f(t) = L_1$, $\lim_{t\to a} g(t) = L_2$, $\lim_{t\to a} h(t) = L_3$

- Then

$\lim_{t\to a} \vec{r}(t) = \langle L_1, L_2, L_3 \rangle$

- e.g. $\lim_{t\to \pi/2} (\cos(2t)\hat{i} - 4\sin(t)\hat{j} + \frac{2t}{\pi}\hat{k})$

① Just plug and play

$L = -\hat{1} - 4\hat{j} + \hat{k}$

- e.g. $\lim_{t\to\infty} \langle e^{-t}, \frac{-2t}{t+1}, \tan^{-1}(t) \rangle$

$\lim_{t\to\infty} e^{-t} = 0$, $\lim_{t\to\infty} \frac{-2t}{t+1} = -2$, $\lim_{t\to\infty} \tan^{-1}(t) = \pi/2$

$\langle 0, -2, \pi/2 \rangle$

- 14.2 Calculus of Vector valued functions.
  - Derivatives and integrals
- Derivative of vector valued functions.

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

If f, g and h are differentiable, then $\vec{r}$ is differentiable.

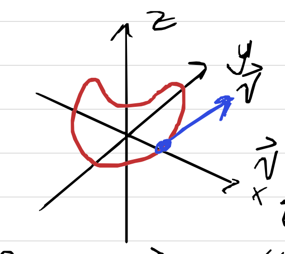
$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

If $\vec{r}'(t) \neq 0$, then the $r'(t)$ is a tangent vector at the point t.

- e.g.

$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 0.4\sin(2t)\hat{k}$

$\vec{r}'(t) = -\sin(t)\,\hat{i} + \cos(t)\hat{j} + 0.8\cos(2t)\hat{k}$



The magnitude of that vector represents the speed. (Rate of change?)

$\vec{v} = (f(a) + f'(a), g(a) + g'(t), h(t) + h'(t))$

Just a state.

- e.g. $\vec{r}(t) = \langle t^3, 3t^2, t^3/6 \rangle$ Find $r'(t)$

$\vec{r}'(t) = \langle 3t^2, 6t, \frac{1}{2}t^2 \rangle$

- No longer setting derivatives to zero.

- The Unit tangent vector is

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

- e.g. $\vec{r}(t) = \langle t^2, 4t, 4\ln(t) \rangle$ Find the unit tanget vector.

① find the derivative

$\vec{r}'(t) = \langle 2t, 4, 4\frac{1}{t} \rangle$

② $|\vec{r}'| = \sqrt{(2t)^2 + (4)^2 + (\frac{4}{t})^2} = 2\sqrt{(t + \frac{2}{t})^2} = 2(t + \frac{2}{t})^2$

- Since the natural log is present, the domain of t is $0 < t$.

③ Substitute

$\vec{T}(t) = \frac{\langle 2t, 4, 4/t \rangle}{2(t + \frac{2}{t})} = \frac{\langle t, 2, 2/t \rangle}{t + 2/t} = \langle \frac{t}{t + 2/t}, \frac{2}{t + 2/t}, \frac{2/t}{t + 2/t} \rangle$

$\langle \frac{t^2}{t^2+2}, \frac{2t}{t^2+2}, \frac{2}{t^2+2} \rangle$

- Let $\vec{u}$ and $\vec{v}$ be vector valued functions and differentiable scalar, on (a, b), and $\vec{c}$ is a constant vector.

$\vec{c}' = \vec{0}$

... Same derivative rules.

dot product

$\frac{d}{dt}(\vec{u}(t) \cdot \vec{v}(t)) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$

Cross product

$\frac{d}{dt}(\vec{u}(t) \times \vec{v}(t)) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$

- Integrals for vector valued functions

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $\vec{R}(t) = \langle F(t), G(t), H(t) \rangle$

Derivative — Antiderivative

- The indefinite integral of $\vec{r}(t)$ is

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}$$

- e.g. $\vec{r}(t) = \langle e^{3t}, \frac{1}{1+t^2}, \frac{-2t}{\sqrt{t^2+4}} \rangle$

③ Take it one step at a time

$\int e^{3t}\,dt = \frac{1}{3}e^{3t} + C_1$

$\int \frac{1}{1+t^2}\,dt = \tan^{-1}(t) + C_2$

$\int -2\frac{t}{\sqrt{t^2+4}}\,dt = -2\sqrt{t^2+4} + C_3$

$\int \vec{r}(t)\,dt = \langle \frac{1}{3}e^{t3}, \tan^{-1}(t), -2\sqrt{t^2+4} \rangle + \vec{C}$

- Vector $\vec{C}$ for final answer.

If Not $\vec{c}$, minus points.

- Definite integral, same ole shit.

$\int_a^b f(t)\,dt$ ...

$\langle F(a) - F(b), G(a) - G(b), H(a) - H(b) \rangle$

Lecture Notes

6 - 3 - 25

- 14.4 Leingths of curves
- Arc leingth of a curve

$$\int_a^b |\vec{r}'(t)|\,dt$$

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$$

- Arc leingth function let $\vec{r}(t)$ describe a smooth curve for $t \geq a$ $(\vec{r}'(t) \neq \vec{0})$

$$s(t) = \int_a^t |\vec{r}'(u)|\,du$$

- e.g. $|\vec{r}'(t)| = 1$

$$s(t) = \int_a^t du = t - a$$

- Re parametrizing a curve function
  → Intead of Inputing the output
  → After an inegral we get
  e.g.

$$\vec{r}(t) = \langle 250 \sin t, 250 \cos t, 100t \rangle$$

$$|r'(t)| = \sqrt{72500}$$

$$s(t) = \int_0^t \sqrt{72500}\,dt = \quad s = t\sqrt{72500}$$

→ Reparametrizing

$$s = t\sqrt{72500}$$

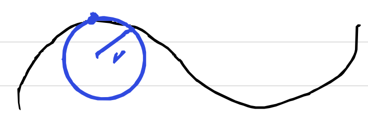
$$t = \frac{s}{\sqrt{72500}}$$

$$\vec{r}(s) = \langle 250 \sin\left(\frac{s}{\sqrt{72500}}\right), 250 \cos\left(\frac{s}{\sqrt{72500}}\right), 100\left(\frac{s}{\sqrt{72500}}\right) \rangle$$

$$|r'(s)| = 1$$

- 14.3 Motion in space
  velosity is $r'(t)$
  speed is $|r'(t)|$
  acceleration $r''(t)$

- 14.5 Curvature and Normal vectors
  - curvature



The smaller the r, the more the curve.

Curvature is the $\frac{1}{r}$ at a particular point.

- Def let $\vec{r}$ be a smooth unparametried curve. If s denotes the arch leingth and $\vec{T}$ is the unit tangent vector, the curvature is κ

$$\kappa(s) = \left|\frac{d\vec{T}}{ds}\right|$$

What if $\vec{r}$ isn't parametrized by s?
→ Use chainrule

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{d\vec{s}}{dt} \quad \Bigg| \quad \frac{ds}{dt} = |r'(t)|$$

- Theorem

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

$\vec{T}$ → Unit tangent vector

- e.g. $\vec{r}(t) = \langle R\cos(t), R\sin(t) \rangle$

$$\vec{r}'(t) = \langle -R\sin(t), R\sin(t) \rangle$$

$$\vec{T}(t) = \frac{r'(t)}{|\vec{r}'(t)|} = \langle -\sin t, \cos t \rangle$$

$$|T'(t)| = 1$$

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{1}{R}$$

- Theorem

$$\kappa = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$$

only for 3 d shapes
— as long as third comp = 0

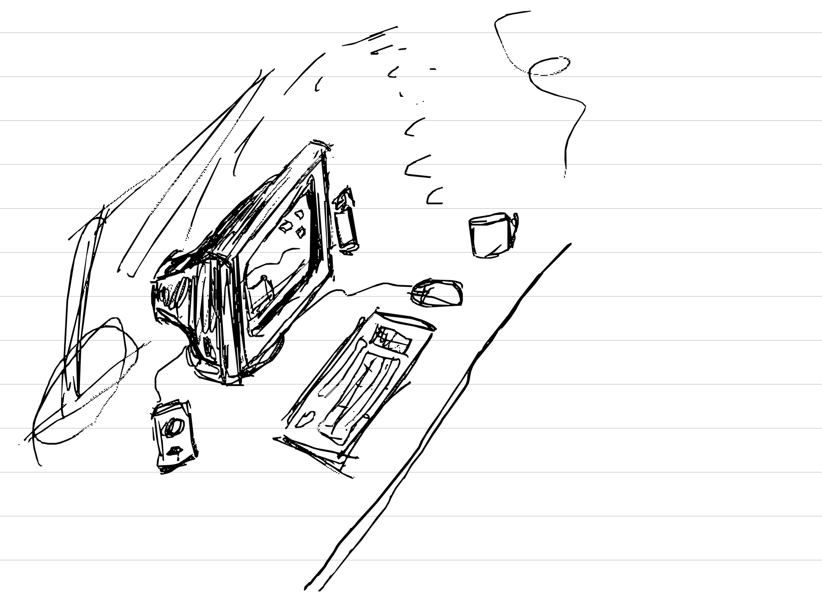
eg $\vec{r}(t) = (t, 2t^2)$

$r'(t) = (1, 4t)$

$r''(t) = (0, 4)$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4t & 0 \\ 0 & 4 & 0 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}\,4$$

$$\frac{4}{(r'(t))} = \kappa = \frac{4}{\sqrt{1+16t^2}^3}$$



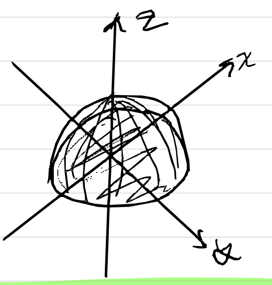
Lecture Notes

- Unit normal vector
  1 — $\vec{T} \cdot \vec{N} = 0$
  2 $\vec{N}$ points inside the curves.
- theorem
  $$\vec{a} = a_n \vec{N} + a_t \vec{T}$$
  where $a_n = K |\vec{v}|^2 = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$
  $$a_t = \frac{d^2 s}{d t^2}$$

a_n is the change of direction
a_t is the change in speed

Multivariable Functions 15.1

- e.g. $f(x,y) = \sqrt{4 - x^2 - y^2}$
  $z = \sqrt{4 - x^2 - y^2}$ square both sides
  $z^2 + x^2 + y^2 = 4$



The domain of a 2 var funct is the area it occupies in the x, y, plane.

— Level curves Def

For a function, $z = f(x,y)$
the curve $f(x,y) = z_0$ is a level curve.
- looking at the graph from the top.

Independent Variables	Explicit	Implicit
1	$y = f(x)$	$0 = F(x,y)$
2	$z = f(x,y)$	$0 = F(x,y,z)$
n	$n = f(x, y, \ldots, n-1)$	$0 = F(x, y, \ldots, n)$

Lecture Notes 20-3-25

- Multivariable limits

- Theorem
  If $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, then f is continuos at (a,b)

- All functions in this class will be continuos

e.g. $\lim_{(x,y)\to(4,1)} \frac{\sin(xy)}{x^2+3y^2} =$ | Just plug in

$= \frac{\sin 4}{9}$

e.g. $\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$ → This is undefined.

Say $y=0$: $\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$

say $x=0$: $\lim_{(x,y)\to(0,0)} \dots = \lim_{y\to 0} \frac{y^2}{y^2} = 1$

Not showing How this is DNE.

Try $y = x$

- e.g. $\lim_{x\to 0} \frac{(2x)^2}{2x^2} = 2$

$\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2+y^4}$ show that the given limit DNE

approach the limit from the line $y = mx$

$\lim_{(x,mx)\to(0,0)} \frac{3x^2}{x^2+y^4} = \lim_{(x,mx)\to(0,0)} \frac{3x^2}{x^2+(mx)^4}$

$\lim_{(x,mx)\to(0,0)} \frac{3\cancel{x^2}}{\cancel{x^2}(1+m^4x^2)} = \lim_{x\to 0} \frac{3}{1+m^4x^2} = 3$

other approach

— Idea combine x^2+y^2 into one expression

say $x = y^2$

$\lim_{(x,y)\to(0,0)} \dots = \lim_{y\to 0} \frac{3y^4}{y^4+y^4} = \frac{3}{2}$

- It's free game to what you like to use for y, x, whatever.

15.3 Derivative on multivariable

- Partial derivatives: when you have mutivariables
- Def Let $z = f(x,y)$

$$\frac{\partial z}{\partial x} = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial z}{\partial x} = f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

— The variable with no h, cancels.

- For derivatives, same as before, but when you take the derivative with respect to say x, treat y as a constant (Same otherway round)

- E.g. Find the two partial derivatives

$g(u,v) = v^7\ln(u^2) + \frac{9}{v^3} - \sqrt[7]{u^4}$

$g_u(u,v) = \frac{2v^7}{u} + 0 - \frac{4}{7}u^{-3/7}$

$g_v(u,v) = 7v^6\ln(u^2) + 27\frac{1}{v^4} - 0$

- E.g. $f(x,y) = \sin(xy) + x^2e^{5y} + 3y^2$
  Find the derivatives

$f_x(x,y) = y\cos(xy) + 2xe^{5y} + 0$

$f_y(x,y) = x\cos(xy) + 5e^{5y}x^2 + 6y$

First order derivatives

- Second order derivatives notation

$(f_x)_x = f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = -y^2\sin(xy) + 2e^{5y}$

$(f_y)_y = -x^2\sin(yx) + 25x^2e^{5y} + 6$

$(f_x)_y = f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$ Order is fliped

$y\cos(xy) + 2xe^{5y} + 0$

$(f_x)_y = f_{xy} = y\frac{\partial f}{\partial y}(\cos(xy)) + \cos(xy)\frac{\partial f}{\partial y}(y) + \dots$

$-y^2\sin(xy) + \cos(xy) + \frac{\partial f}{\partial y}\ 10xe^{5y}$

$= \cos(xy) - y^2\sin(xy) + 10xe^{5y}$

$(f_y)_x = \frac{\partial f}{\partial x}[x\cos(xy) + 5e^{5y}x^2 + 6y] =$

! They both equal the same

- Clurouse theorem $(f_y)_x = (f_x)_y$

- e.g. $f(x,y) = \begin{cases} xy\frac{x^2-y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Lecture Notes

10/4/25

If f has a local maximum or a local minimum at (a, b) and f_x and f_y exist at (a, b) then

$$f_x(a, b) = f_y(a, b) = 0$$

- An interior point (a, b) in the domain of f is a critical point if either
  1) $f_x(a, b) = f_y(a, b) = 0$
  2) at least one of f_x or f_y is undefined at (a, b)

- e.g. $f(x,y) = xy(x-2)(y+3)$

$f_x = y(y+3)(2x-2)$

$f_y = x(x-2)(2y+3)$

Solve

$$\begin{cases} y(y+3)(2x-2) = 0 \\ x(x-2)(2y+3) = 0 \end{cases}$$

Pick one of them

$y(y+3)(2x-2) = 0$

$y = \{0, -3\}$ $x = \{1\}$

Find all possible solutions to the equation

Pick the found values into the other equation and verify

$y = 0$

$x(x-2)(2(0)+3) = 0$

$x(x-2)(3) = 0$

For this equation we must have $(0,0)$ $(2,0)$ (the x that satisfy the solution)

$y = -3$

$x(x-2)(-3)$ crit points $(0, -3)$ $(2, -3)$

$x = \{0, 2\}$

$x = 1$

$-2(2y+3) = 0$ crit points $(1, -3/2)$

$y = -3/2$

→ Critical points

$(0,0), (2,0), (0,-3), (2,-3)\ (1, -3/2)$

→ Find if it's a max or a min

- Using 2 derivative test

Let F be a function that's differentiable at some point P. If every open disk centered at P contains the point (x_1, y_1) and (x_2, y_2) such that

$$f(x_1, y_1) > f(a, b) \quad \text{and}$$
$$f(x_2, y_2) > f(a, b)$$

then (a, b) is a saddle point.

→ Second Derivative test

Suppose the second partial derivatives of f are continuous on an open disk centered at (a, b) and $f_x(a, b) = f_y(a, b) = 0$

let $D(x, y) = f_{xx}(x,y) f_{yy}(x,y) - (f_{xy}(x,y))^2$

1) if $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum at $(a,b$
2) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then f has a local min at (a,b)
3) If $D(a,b) < 0$, then f has a saddle point at (a,b)
4) If $D(a,b) = 0$, then the te test is inconclusive

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \text{Hessia}$$

$f_x = y(y+3)(2x-2)$

$f_y = x(x-2)(2y+3)$

$f_{xx} = 2y(y+3)$

$f_{yy} = 2x(x-2)$

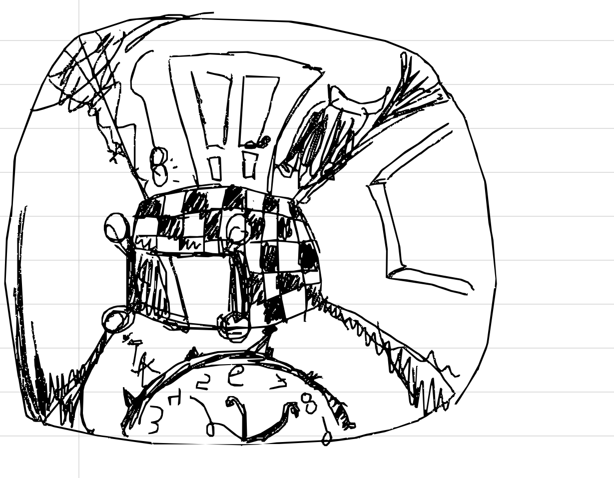
$f_{xy} = (2x-2)(2y+3)$

$D = 4xy(y+3)(x-2) - ((2x-2)(2y+3)^2$

→ crit

$(0,0), (2,0), (0,-3), (2,-3)\ (1, -3/2)$

Point	D	f_{xx}	Conclution
(0, 0)	−36		Saddle
(2, 0)	−36		Saddle
(0, −3)	−		Sadle
(2, 0)	−		Sadle
(2, −3)	−		saddle
(1, −3/2)	4	−	Maximum



Lecture Notes

e.g. Find the max/min in a clossed interval

$f(x,y) = xy - 8x - y^2 + 12y + 160$

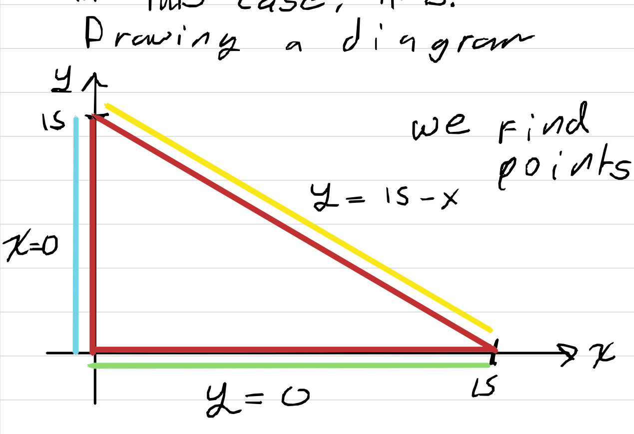
On the region

$R = \{(x,y) \mid 0 \le x \le 15,\ 0 \le 15 - x\}$

$f_x(x,y) = y - 8 \quad = 0 \rightarrow y = 8$
$f_y(x,y) = x - 2y + 12 \quad = 0 \rightarrow x - 16 + 12 = 0$
$x = 4$

Critical point at $(4,8)$
— Must verify that the critical point is in the Bounded region
- In this case, it is.

Drawing a diagram



we find candidate critical points

Using the line $y = 0$

$g_1 = f(x,0) = -8x + 5$

Using the line $x = 0$
$g_2 = f(0,y) = -y^2 + 12y + 5$

Using the line $y = 15 - x$

$g_3 = f(x, 15 - x) = x(15 - x) - 8x - (15 - x)^2 +$

Checking the g's

$g_1(x) = -8x + 5$
$g_1'(x) = -8$

set to zero to find the critical point
$-8 \neq 0$

$g_2(y) = -y^2 + 12y - 5$
$g_2'(y) = -2y + 12$

$-2y + 12 = 0$
$y = 6$
Max / min at $(0, 6, 15)$

$g_3(x) = f(x, 15 - x) = -2x^2 + 25x + 115$
$g_3'(x) = -4x + 25$
$x = \frac{25}{4}$

(x,y)	$f(x,y)$	
$(4,8)$	192	
$(0,0)$	160	
$(15,0)$	40	min
$(0,6)$	196	max
$(0,15)$	115	
$(25/4, 35/4)$	$\frac{1545}{8} = 193.125$	

ANSwer

Min at $(15,0)$
Max at $(0,6)$

15.8 Lagrange Multipliers.

Theorem: Let f be a differentiable function in some region of $\mathbb{R}^2$ that contains the smooth curve C given by $g(x,y) = 0$. Assume f has some local extreme at a point $P(a,b)$. Then $\nabla f(a,b)$ is parallel to $\nabla g(a,b)$. $\nabla f(a,b) = \lambda \nabla g(a,b)$ where λ is a Lagrange multiplier.

- e.g. $f(x,y) = 81x^2 + y$
  constraint equation
  $4x^2 + y^2 = 9$
- e.g. $f(x,y) = x^2 + y^2 + 2$
  Subject to $x^2 + xy + y^2 - 4 = 0$

$g(x) = x^2 + xy + y^2 - 4$

$f_x(x,y) = 2x \quad ; \quad f_y(x,y) = 2y$
$g_x(x,y) = 2x + y \quad ; \quad g_y(x,y) = 2y + x$

$$\begin{cases} 2x = \lambda(2x + y) \\ 2y = \lambda(2y + x) \\ x^2 + xy + y^2 - 4 = 0 \end{cases}$$

• e.g. $f(x,y) = x^2 y$ subject to $x^2 + y^2 = 9$

$f_x(x,y) = 2xy \qquad f_y(x,y) = x^2$
$g_x(x,y) = 2x \qquad g_y(x,y) = 2y$

$$\begin{cases} 2xy = 2\lambda x \\ x^2 = 2\lambda y \\ x^2 + y^2 = 9 \end{cases}$$

$2xy = 2\lambda x$
$2xy - 2\lambda x = 0$
$2x(y - \lambda) = 0$

$$\begin{cases} x = 0 \\ y = \lambda \end{cases}$$

$\{x = 0\} \rightarrow y^2 = 9 \quad y = \pm 3$

$(0,3), (0,-3)$

$\{\lambda = y\} \quad x^2 = 2y^2$

$x^2 + y^2 = 9$

$2y^2 + y^2 = 9$

$y^2 = 3$
$y = \pm\sqrt{3}$

$x^2 + 3 = 9$
$x^2 = 6$
$x = \pm\sqrt{6}$

$(\sqrt{6}, \sqrt{3}), (\sqrt{6}, -\sqrt{3}), (-\sqrt{6}, \sqrt{3}), (-\sqrt{6}, -\sqrt{3})$

(x,y)	$f(x,y)$	
$(0,-3)$	0	
$(0,3)$	0	
$(\sqrt{6}, \sqrt{3})$	$6\sqrt{3}$	Max
$(\sqrt{6}, -\sqrt{3})$	$-6\sqrt{3}$	min
$(-\sqrt{6}, \sqrt{3})$	$6\sqrt{3}$	Max
$(-\sqrt{6}, -\sqrt{3})$	$-6\sqrt{3}$	min

• e.g. $f(x,y,z) = xyz$ Subject to $x + 9y^2 + z^2 = 4$
for $x \ge 0$

$f_x(x,y,z) = yz \qquad g_x(x,y,z) = 1$
$f_y(x,y,z) = xz \qquad g_y(x,y,z) = 18y$
$f_z(x,y,z) = xy \qquad g_z(x,y,z) = 2z$

$$\begin{cases} yz = \lambda \\ xz = \lambda 18y \\ xy = 2\lambda z \\ x + 9y^2 + z^2 = 4 \end{cases}$$

$xyz = x\lambda$
$xyz = 18\lambda y^2$
$xyz = 2\lambda yz$

$18\lambda y^2 = x\lambda$
$18\lambda y^2 - \lambda x = 0$
$\lambda(18y^2 - x) = 0$
$\lambda = 0, \quad 18y^2 = x$

$18y^2 + 9y^2 + z^2 = 4$ $y?$

$18\lambda y^2 = 2\lambda yz$
$18\lambda y^2 - 2\lambda yz = 0$
$\lambda y(18y - 2z) = 0$
$\lambda = 0, y = 0, 9y = z$

$18y^2 + 9y^2 + 9y = 4$
$y = \pm 1/3$
$x = 18(\frac{1}{9}) = 2, \quad z^2 = 9(1/9) = 1 = \pm 1$
$(2, 1/3, 1) \ (2, 1/3, -1)$
$(2, -1/3, 1) \ (2, -1/3, -1)$

$\lambda = 0$
$yz = 0 \qquad y = 0$ or $z = 0$
$xz = 0 \qquad x = 0$ of $z = 0$
$xy = 0 \qquad x = 0$ or $y = 0$

for $y = 0 \quad x = 0 \quad$ using $x^2 + y^2 \ldots$
$z = 4$
$(0,0,2)(0,0,-2)$

for $x = 0 \quad z = 0$
$9y^2 = 4$
$y = \pm 2/3$
$(0, 2/3, 0), (0, -2/3, 0)$

for $y = 0, z = 0$
$x = 4$
$(4,0,0)$

Plug and Plug

Lecture Notes 17/4/25

$x^2 y$ subject to $x^2+y^2=9$

$f_x = 2xy \qquad g_x = 2x$
$f_y = x^2 \qquad g_y = 2y$

$$\begin{cases} 2xy = 2\lambda x \\ x^2 = 2\lambda y \\ x^2+y^2 = 9 \end{cases}$$

$2\lambda y + y^2 = 9$
$y^2 + 2\lambda y - 9 = 0$

$2x^2 y = 2\lambda x^2$
$2(2\lambda y)y = 2\lambda(2\lambda y)$
$4\lambda y^2 = 4\lambda^2 y$
$\lambda y^2 - \lambda^2 y = 0$
$\lambda y(y-\lambda) = 0$
$\lambda y = 0$
$y - \lambda = 0$

$y = \lambda, 0 \qquad \lambda = y, 0$

$y=0$
$x^2 + 0 = 9$
$x = \pm 3$

$(3,0)$
$(-3,0)$

$y=\lambda$ $\quad x^2 = 2y^2 \leftarrow ?$ maybe?

6

$2y^2 + y^2 = 9$
$3y^2 = 9$
$y^2 = 3$
$y = \pm\sqrt{3}$

$x^2 + 3 = 9$
$x = \pm\sqrt{6}$

$\{(-\sqrt6, -\sqrt3), (\sqrt6, -\sqrt3), (-\sqrt6, \sqrt3), (\sqrt6, \sqrt3)\}$

Integral again

A function f derived on a rectangular region R in the xy plane is integrable if

$\lim_{\Delta A_k \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k$ Exists for all partitions of R and choises of (x_k^*, y_k^*)

$$\lim_{k \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k = \iint_R f(x,y)\, dA$$

Iterated Integral

$$\int_0^2 \left(\int_0^1 4xy\, dx\right) dy$$

$$\int_0^2 y(2)\, dy = 4\int_0^1 y\, dy =$$

$$2(\tfrac{1}{2}y^2)\Big|_0^1 = 2$$

---

e.g.

$$\int_1^3 \int_1^2 3x^2 + 4y^2\, dy\, dx$$

$$\int_1^3 3x^2 y + y^4 \Big|_1^2\, dx$$

$$\int_1^3 3x^2 + 15\, dx$$

$$x^3 + 15\Big|_1^3 = (27+45) - (1+15)$$
$$72 - 16 = 56$$

Theorem Fubini's Theorem

Let f be continuos over the region $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dx\, dy = \int_b^c \int_a^b f(x,y)\, dy\, dx$$

e.g $\displaystyle\int_0^1 \int_0^{\pi/2} uv\cos(u^2 v)\, dv\, du$

$$\int_0^{\pi/2} \int_0^1 uv\cos(u^2 v)\, du\, dv$$

$w = u^2 v$
$\frac{dw}{du} = 2uv$

$dw = 2uv\, du$
$\frac{1}{2} dw = uv\, du$

$$\int_0^{\pi/2} \frac{1}{2} \int_a^b \cos(w)\, dw\, dv$$

$$\int_0^{\pi/2} \frac{1}{2} \int_0^1 \sin(u^2 v)\, du\, dv$$

$$\frac{1}{2}\int_0^{\pi/2} \sin(v)\, dv$$

$$-\frac{1}{2}\cos(v)\Big|_0^{\pi/2}$$

$$\boxed{-\tfrac{1}{2}}$$

16.2- Double integrals over general regions

e.g. $\displaystyle\int_0^1 \int_0^{2x} 15xy^2\, dy\, dx$

Limits of the last integral have to be numbers
The bounds of the integral have be different to what we are integrating with respecto.

$$\int_0^1 \int_0^{2x} 15xy^2\, dy\, dx$$

$$\int_0^1 5xy^3\Big|_0^{2x} dx$$

$$40\int_0^1 x^4\, dx$$

$$8x^5\Big|_0^1 = 8$$

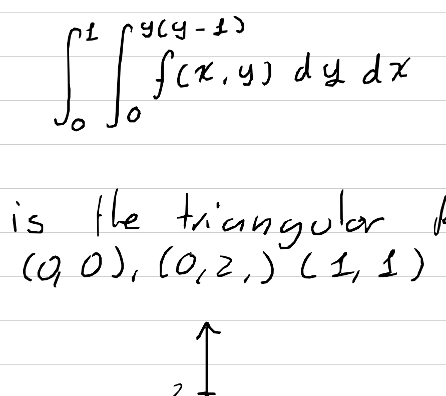
Def: let $R = \{(x,y) \mid a \le x \le b,\ g(x) \le y \le h(x)\}$
If f is continuous on R then

$$\iint_R f(x,y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\, dx\, dy$$

Let $R = \{(x,y) \mid c \le y \le d,\ g(y) \le x \le h(y)\}$

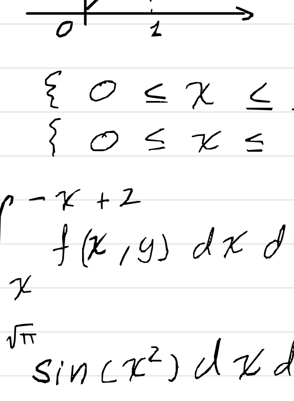
$$\iint_R f(x,y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\, dy\, dx$$

e.g. $R\{(x,y) \mid 0 \le x \le y(1-y)\}$



$$\int_0^1 \int_0^{y(y-1)} f(x,y)\, dy\, dx$$

e.g. R is the triangular Region with vertices (0,0), (0,2,) (1,1)

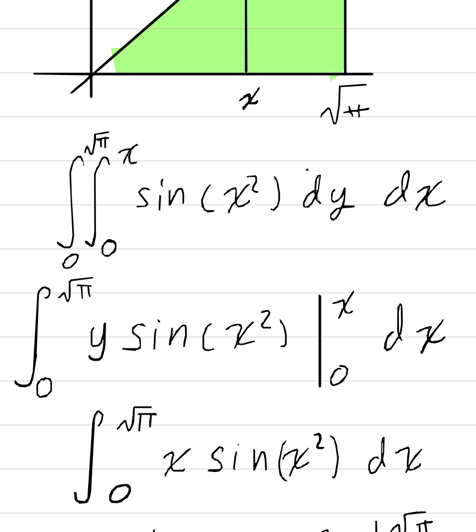


$y = -x+2 \qquad \{0 \le x \le 1\}$
$y = x \qquad \{0 \le x \le 1\}$

$$\int_0^1 \int_x^{-x+2} f(x,y)\, dx\, dy$$

e.g. $\displaystyle\int_0^{\sqrt\pi} \int_y^{\sqrt\pi} \sin(x^2)\, dx\, dy$

$y \le x \le \sqrt\pi$
$0 \le y \le \sqrt\pi$



$$\int_0^{\sqrt\pi} \int_0^x \sin(x^2)\, dy\, dx$$

$$\int_0^{\sqrt\pi} y\sin(x^2)\Big|_0^x dx$$

$$\int_0^{\sqrt\pi} x\sin(x^2)\, dx$$

$$-\frac{1}{2}\cos(x^2)\Big|_0^{\sqrt\pi}$$

$$\frac{1}{2} + \frac{1}{2} = 1$$

Lecture Notes 21/4/25

Find the volume enclosed by $x^2+y^2=4$ the plane $z=3-x$ and $z=x-3$

$$\iint (3-x)-(-3+x)\, dA$$

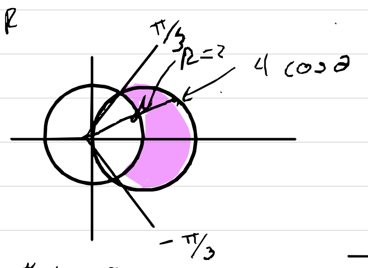
$$\iint 6-2x \, dA$$

$$\int_0^{\pi/2}\int_0^{2} (6-2r\cos\theta)\, r\, dr\, d\theta$$

Theorem Let f be continuos over the region $R=\{(r,\theta) \mid 0\le g(\theta)\le r\le h(\theta),\ \alpha\le\theta\le\beta\}$ where $0<\beta-\alpha\le 2\pi$

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta\int_{g(\theta)}^{h(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

- e.g. $\iint_R g(r,\theta)\, dA$ R is inside of $r=4\cos\theta$ $r=2$



$4\cos\theta = 2$

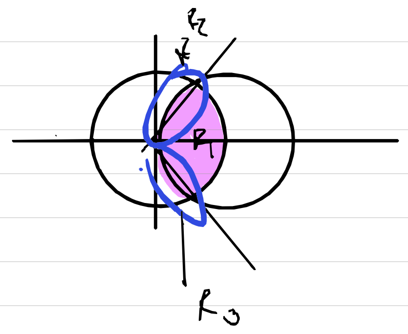
$\cos\theta = 1/2$

$\theta=\{-\frac{\pi}{3}, \frac{\pi}{3}\}$

$$\int_{-\pi/3}^{\pi/3}\int_2^{4\cos\theta} g(r,\theta)\, r\, dr\, d\theta$$

→ Careful with integration bounds

Intersection of $r=4\cos\theta$ $r=2$

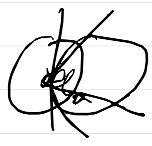


The flaps!

$$\iint g(r,\theta)\, A = \iint_{R_1} g(r,\theta)\, dA + \iint_{R_2} g(r,\theta)\, dA + \iint_{R_3} g(r,\theta)\, dA$$

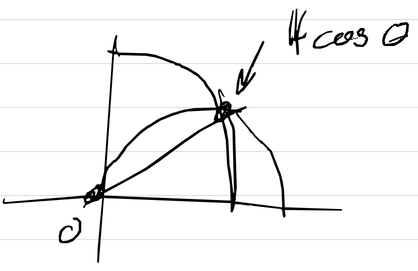
Closer look to R_1

$$\int_{-\pi/3}^{\pi/3}\int_0^{2} g(r,\theta)\, dA$$



Closer look to R_2

$$\int_{\pi/3}^{\pi/2}\int_0^{4\cos\theta} g(r,\theta)\, dA$$



$4\cos\theta = 0$

$\cos\theta = 0$

$\theta = \pi/2$

Closer R_3

$$\int_{-\pi/2}^{-\pi/3}\int_0^{4\cos\theta} g(r,\theta)\, dA$$

$r\cos\theta + i r\sin\theta = re^{i\theta}$

$i\theta = -x^2$

$\theta = i^3 - x^2$

$\theta = ix^2$

e.g. $\int_{-\infty}^{\infty} e^{-x^2}\, dx$

$$\boxed{\int_{-\infty}^{\infty} \cos(ix^2) + i\sin(ix^2)\, dx}$$

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy$$

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

$$2\pi\int_0^{\infty} e^{-r^2}\, r\, dr$$

$$2\pi \lim_{b\to\infty}\int_0^b e^{-r^2}\, r\, dr = -\pi \lim_{b\to\infty} e^{-r^2}\Big|_0^b = I^2 = \pi$$

$$I = \sqrt{\pi}$$

Lecture Notes

$y = 9 - x^2$ and $2x^2 + 3z^2 = y$

$$9 - x^2 = 2x^2 + 3z^2$$
$$9 = 3x^2 + 3z^2$$
$$3 = x^2 + z^2 \qquad x = \pm\sqrt{3 - z^2}$$

$$\iiint_R dv = \int_{-\sqrt{3-z^2}}^{\sqrt{3-z^2}} \int$$

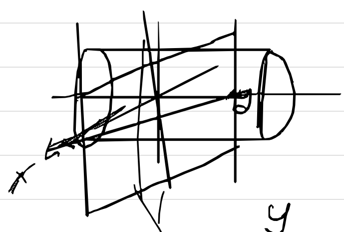
Order change

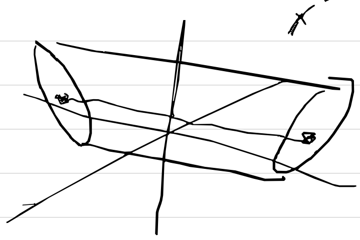
e.g. $\int_0^5 \int_{-1}^{0} \int_0^{4x+4} dy\,dx\,dz \to dz\,dx\,dy$

$$\int_0^4 \int_{\frac{y}{4}-1}^{0} \int_0^5 dz\,dx\,dy$$

e.g. = Set up a triple integral to find the volume of the region bounded by $x^2 + 4z^2 = 4$, and $y = 3 - x$, $y = x - 3$

$$\frac{x^2}{4} + z^2 = 1$$





$$\iint \int_{x-3}^{3-x} dy\,dx\,dz$$

$$x = \pm\sqrt{4 - 4z^2}$$

$$\int_{-1}^{1} \int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}} \int_{x-3}^{3-x} dy\,dx\,dz$$

Lecture Notes

- exam Review

$f(x,y) = y\, e^{(x^2-y^2)}$

$f_x(x,y) = y\, 2x\, e^{x^2-y^2}$

$f_y(x,y) = e^{x^2-y^2} + y(-2y\, e^{x^2-y^2})$

$= e^{x^2-y^2} - 2y^2 e^{x^2-y^2}$

$y\, x\, e^{x^2-y^2} = 0$

Plug critical points in to discriminant

$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - (f_{xy}(x,y))^2$

Process

1 Find derivatives
$f_x, f_y, f_{xx}, f_{yy}, f_{xy}$

2 Set $f_x = 0$ and $f_y = 0$

3 Plug the x ~~and~~ y found from the derivative into

4 Points go onto the discriminant
$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - (f_{xy}(x,y))^2$

5 If the determinant is
$D(a,b) > 0$ and $f_{xx}(a,b) < 0$, (a,b) is a max
$D(a,b) > 0$ and $f_{xx}(a,b) > 0$, (a,b) is a max
$D(a,b) < 0$, (a,b) is a saddle point
$D(a,b) = 0$, inconclusive.

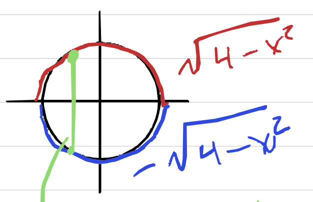
Chain rule

$z = v^3 - 2u^2 v \quad u = \cos y, \quad v = \sin x$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}$

$(3v^2 - 2u^2)(\cos(x)) - (4uv)(0)$

e.g. $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \cos(x^2+y^2)\, dy\, dx$



Integrating from blue to red.

Turn into Polar

$\int_0^{2\pi}\int_0^{2} \cos(r^2)\, r\, dr\, d\theta$

$u = r^2$

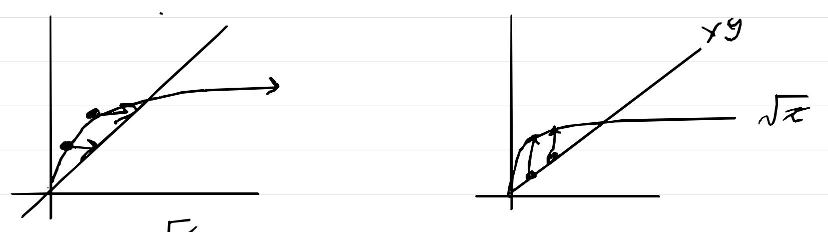
$\frac{du}{dr} = 2r$

$\int_0^{2}\int_a^{b} \frac{1}{2}\cos(u)\, du\, d\theta$

$\frac{1}{2}\int_0^{2\pi} -\sin(r^2)\Big|_0^2$

$\pi(\sin(4) - \sin(0)) = \pi\sin(4)$

e.g. $\int_0^1\int_{y^2}^{y} dx\, dy$



$\int_0^1\int_x^{\sqrt{x}} dy\, dx$

e.g. $f(x,y) = x^2 e^{2xy}$ at $(1,0)$

Find the gradiant

$f_x(x,y) = 2y x^2 e^{2xy} + 2x e^{2xy}$

$f_y(x,y) = 2x^3 e^{2xy} \qquad (1,0)$

Calculating the gradiant, plug in P into f_x and f_y

$\nabla f = (f_x, f_y)$

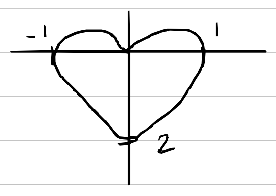
$\nabla f(0,1) = (f_x(0,1), f_y(0,1)) = \langle 2, 2\rangle = \vec{v}$

$\bar{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle \qquad |\vec{v}| = \sqrt{8}$ (greatest rate of change)

$f(x,y,z)$

e.g. Find the area inside $r = 1 - \sin\theta$

$\iint_R dA$



$\int_0^{2\pi}\int_0^{1-\sin\theta} r\, dr\, d\theta$

$\int_0^{2\pi} \frac{1}{2}r^2\Big|_0^{1-\sin\theta} d\theta = \frac{1}{2}\int_0^{2\pi}(1-\sin\theta)^2\, d\theta$

$\int \frac{1-\cos 2\theta}{2}$

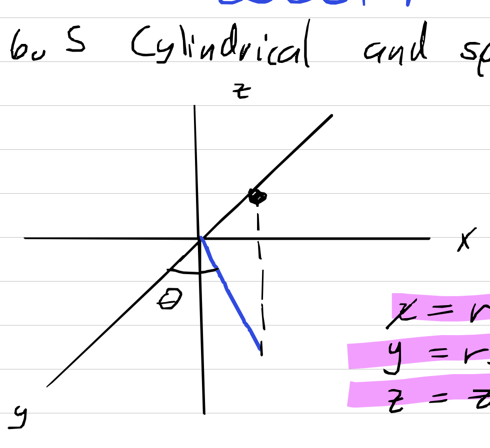
$\frac{1}{2}\int_0^{2\pi} 1 - 2\sin\theta + \sin^2\theta\, d\theta \qquad \frac{1}{2}\int 1 - \cos 2\theta\, d\theta \qquad \frac{1}{2}(\theta - \frac{1}{2}\sin 2\theta)$

$\frac{1}{2}\left(\theta + 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi}$

$\frac{1}{2}(2) - \frac{1}{2}(2\pi + 2 + \pi) = \boxed{\frac{3\pi}{2}}$

Lecture Notes 6/5/25

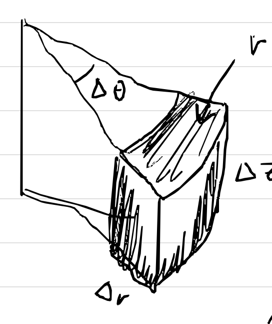
- 16.5 Cylindrical and spherical coordinates.



$(x, y, z) \Rightarrow (r, \theta, z)$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
} Change of coords

$\iiint_D f(x,y,z)\,dv$ we want to write in polar



$r\,\Delta r\,\Delta\theta$

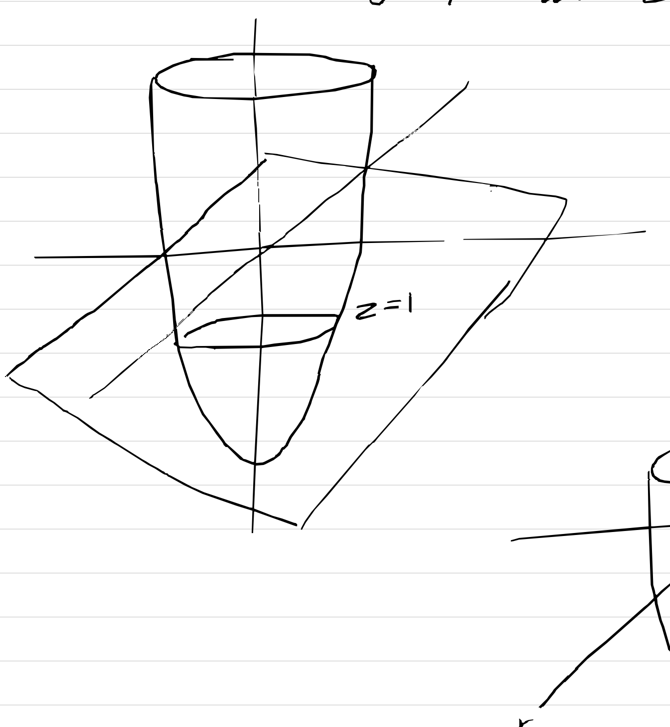
The volume ΔV is the area of the base times the height $r\Delta r\Delta\theta\Delta z = \Delta V$

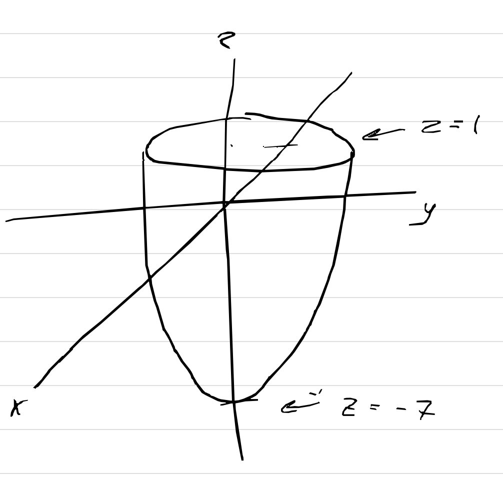
$$\iiint_D f(x,y,z)\,dv = \lim_{n\to\infty}\sum_{k=1}^{\infty} f(x^*, y^*, z^*)\, r_k^*\Delta r_k \Delta\theta_k^*$$

$$= \iiint f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

The important thing about the integration on cilinders is that the order goes $r\,dz\,dr\,d\theta$

e.g. $\iiint_D 4xy\,dv$ D is the region bounded by $z = 2x^2 + 2y^2 - 7$ and $z = 1$





$z = 1$
$z = 2r^2\cos^2\theta + 2r^2\sin^2\theta - 7$
$z = 2r^2 - 7$

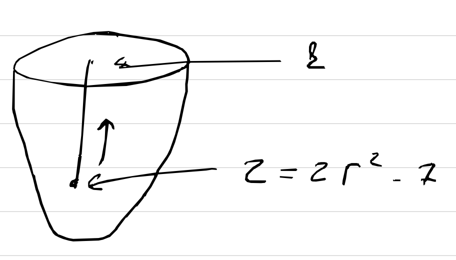
$$\int_0^{2\pi}\int_0^2\int_{2r^2-7}^1 4r^2\cos\theta\sin\theta\, r\,dz\,dr\,d\theta$$

$2r^2 - 7 = 1$
$2r^2 = 8$
$r^2 = 4$
$r = 2$

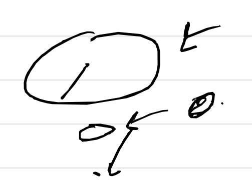
Set up consistently to get r positive

The oter two integrals are simply set like prevous polar integrals

$$\int_0^{2\pi}\int_0^2\int_{2r^2-7}^1 2r^2\sin 2\theta\, dr\,d\theta$$



Then to continue figure out the integral of the rings



$$\int_0^{2\pi}\int_0^2 4r^3\cos\theta\sin\theta\, z\Big|_{2r^2-7}^{1}\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^2 2r^3\sin 2\theta\,(1 - 2r^2 + 7)\,dr\,d\theta$$

$$\int_0^{2\pi}\sin 2\theta\int_0^2 2r^3(8 - 2r^2)\,dr\,d\theta$$

$$\int_0^{2\pi}\sin 2\theta\int_0^2 16r^3 - 4r^5\,dr\,d\theta$$

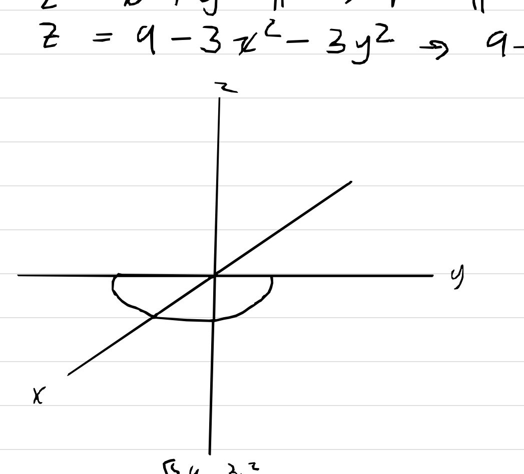
$$\int_0^2 \dots = 0$$

e.g. $D = \{(x, y, z) \mid x^2 + y^2 - 11 \le z \le 9 - 3x^2 - 3y^2,\ -\sqrt{5 - x^2} \le y \le 0$

Density at an object at a point

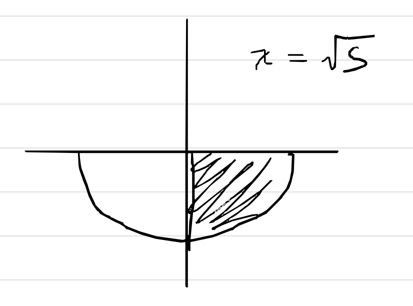
- Convert to cylindrical

$z = x^2 + y^2 - 11 \to r^2 - 11 = z$
$z = 9 - 3x^2 - 3y^2 \Rightarrow 9 - 3r^2 = z$



$y = -\sqrt{5 - x^2}$

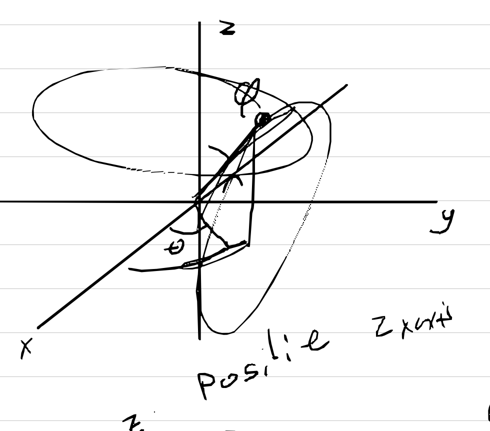
$x = \sqrt{5}$



$$\int_{-\frac{\pi}{2}}^{0}\int_0^{\sqrt{5}}\int_{r^2-11}^{9-3r^2}(2r\cos\theta - 2r\sin\theta)\,r\,dz\,dr\,d\theta$$

see how the lower and upper bounds for z are given.

- Spherical Coordinates

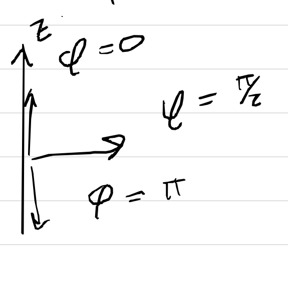


$(x, y, z) \to (\rho, \theta, \varphi)$

$\rho > 0$

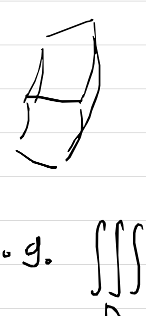
$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi$



$x = \rho\sin(\varphi)\cos(\theta)$
$y = \rho\sin(\varphi)\sin(\theta)$
$z = \rho\cos(\varphi)$

$\rho^2 = x^2 + y^2 + z^2$



$dv = \rho^2\sin(\varphi)\,d\rho\,d\theta\,d\varphi$

e.g. $\iiint_D (10xz + 3)\,dv$ where D is the region inside $x^2 + y^2 + z^2 = 16$ and above $z = 0$

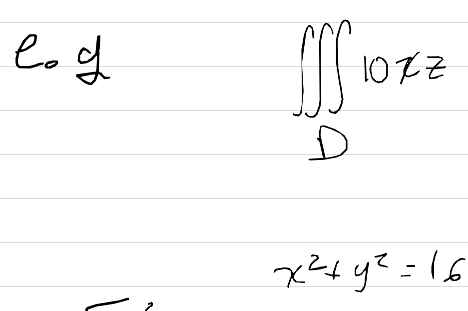
$$\iiint (10(\rho\sin\varphi\cos\theta)(\rho\cos\varphi) + 3)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$$\iiint (10\rho^2\sin\varphi\cos\varphi\cos\theta + 3)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$x^2 + y^2 + z^2 = 16 \to \rho^2 = 16$, $\rho = 4$ ← $z = 0$

$$\int_0^{\pi/2}\int_0^{2\pi}\int_0^4 (10\rho^2\sin\varphi\cos\varphi\cos\theta + 3)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

Bc $z > 0$ only top of the sphere



- e.g. $\iiint_D 10xz + 3\,dv\quad dz\,dy\,dx$

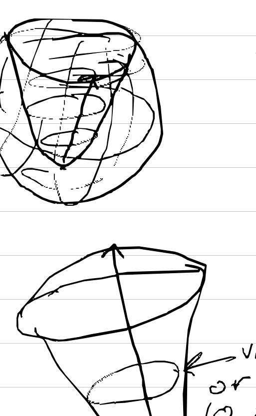
$z = \sqrt{16 - x^2 + y^2}$
$z = 0$

$x^2 + y^2 = 16$

$$\int_{-4}^{4}\int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}}\int_0^{\sqrt{16x^2+y^2}}(10xz + 3)\,dz\,dx\,dy$$

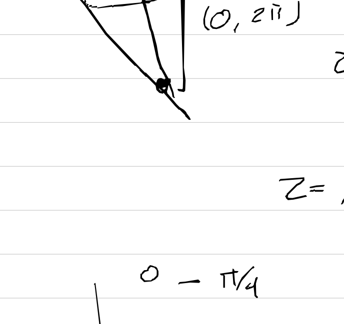
- To pick an integration technique look at the bounds. Circle cylindrical, sphere, sphereccal

- e.g. $\iiint_D dv$ inside both $x^2 + y^2 + z^2 = 1$ ← Sphere and $z = \sqrt{x^2 + y}$ ← Kone



The center to the edge of the region

$$\int_0^{\pi/4}\int_0^{2\pi}\int_0^1 \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$



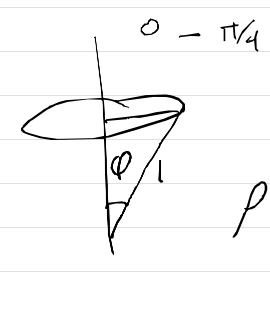
$x^2 + y^2 + z^2 = 1$

$z^2 = x^2 + y^2$

$2x^2 + 2y^2 = 1$
$x^2 + y^2 = \frac{1}{2}$
↑
circle

$z = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad \pi/4$

$0 - \pi/4$



$(\sim, \sim, \sqrt{\frac{1}{2}})$

$\rho = 1$

$z = \rho\cos(\varphi)$
$\frac{\sqrt{2}}{2} = \cos\varphi$
$\varphi = \pi/4$

Lecture Notes 8/5/25

- **17.1 Vector Fields**

a point is given, then a vector is created using the values of that point at that point

$\bar{F}(a,b) = \langle g(a), h(b) \rangle$ @ (a,b)

- **Gradient Field**

Let φ be a differentiable region in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\nabla\varphi = \bar{F}$, $\bar{F}$ is a radient field and φ is the potential function to $\bar{F}$.

$$\varphi(x,y) = \tan^{-1}(x,y)$$

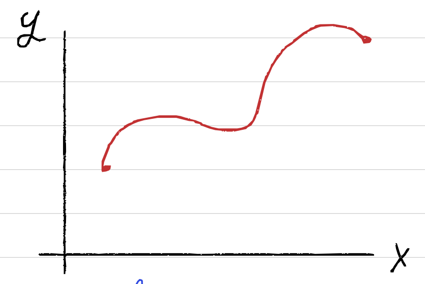
$$\varphi_x(x,y) = \frac{y}{1+(xy)^2}$$

$$\varphi_y(x,y) = \frac{x}{1+(xy)^2}$$

$$\bar{F} = \left\langle \frac{y}{1+(xy)^2}, \frac{x}{1+(xy)^2} \right\rangle$$

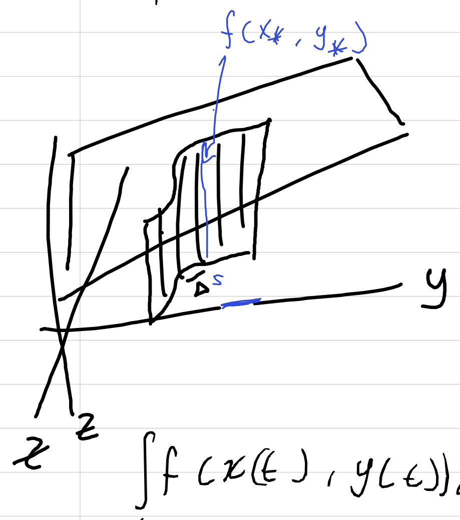
- **17.2 Line Integra**

– Scalar line integral.



$C = \bar{r}(t) = \langle x(t), y(t) \rangle$

$f(x,y)$



$$\int_C f(x(t), y(t))\, ds$$

$$\int_C f(x(t), y(t))\,ds = \lim_{n\to\infty} \sum_{k=1}^{\infty} f(x(t_k^*), y(t_k^*))\,\Delta s$$

- Evaluating line integrals $\int_C f\,ds$

$C = \bar{r}(t) \quad a \le t \le b$

Arch leingh

$$s(t) = \int_a^t |\bar{r}'(u)|\,du$$

$$s'(t) = \bar{r}'(t) = \frac{ds}{dt}$$

$$ds = |\bar{r}'(t)|\,dt$$

$$\int_C f\,ds = \boxed{\int_a^b f(x(t), y(t))\,|\bar{r}'(t)|\,dt}$$

- e.g. $\int_C xy\,ds$, C is $\bar{r}(t) = \langle \cos(t), \sin(t) \rangle$

$0 \le t \le 2\pi$

$r'(t) = \langle -\sin(t), \cos(t) \rangle$

$|r'(t)| = 1$

$$\int_0^{2\pi} \cos(t)\sin(t)\,dt$$

$$\frac{1}{2}\int_0^{2\pi} \sin(2t)\,dt = 0$$

- e.g. $\int_C (x+y+z)\,ds \quad 0 < t < \pi$

$\bar{r}(t) = \langle 2\cos(t), 0, 2\sin(t) \rangle$

$|r'(t)| = 2$

$$\int_C x+y+c\,ds = \int_0^{\pi} (2\cos t + 2\sin t)\,2\,dt$$

$$4\int_0^{\pi} \cos t + \sin t\,dt$$

$$4(\sin t - \cos t)\Big|_0^{\pi}$$

$$(4) - (-4) = 8$$

Solving steps
1 – $|\bar{r}(t)|$
2 – Write integral
3 – Solve

Bouns $\int$ are bounds of t

- **Line Integrals of vector fields**

Line integral of a vector field $\bar{F}$ containing a smooth C is aparametized curve. Let $\bar{T}$ be the unit tangent vector at each point of C consistent whit the orientation

The line integral of F over C is

$$\int_C \bar{F}\cdot\bar{T}\,ds$$

$C: \bar{r}(t) \quad a \le t \le b$

$$\bar{T} = \frac{r'(t)}{|r'(t)|}$$

$$ds = |r'(t)|\,dt$$

$$\boxed{\int_a^b \bar{F}\cdot r'(t)\,dt}$$

Lecture Notes

- Line integral of a vector field

$$\int_C \bar{F}\cdot\bar{T}\,ds = \int \bar{F}\cdot d\bar{r}$$

$$\int_C f\,dx + g\,dy + h\,dz$$

$$\int_a^b \bar{F}\cdot\bar{r}'(t)\,dt$$

- e.g. $\bar{F} = \langle x, y\rangle$, $\bar{r}(t) = \langle 4t, t^2\rangle$, $0 \le t \le 1$

$$\int_C \bar{F}\cdot\bar{T}\,ds \quad \text{use} \quad \int_a^b \bar{F}\cdot\bar{r}'(t)\,dt$$

$r'(t) = \langle 4, 2t\rangle$

Find

$\bar{F}(\bar{r}(t)) = \langle 4t, t^2\rangle$

$$\int_0^1 16t + 2t^3\,dt$$

$8t^2 + \frac{1}{2}t^4 \Big|_0^1$ $\qquad 2^2 + 3^2 =$

$8 + \frac{1}{2} = \underline{17}$ $\qquad a^2 + (a+1)^2 + (a+2)^2 = n^2$

$\sum_{n=0}^{\infty} (a+n)^2$

- e.g. $\bar{F} = \langle e^{2x}, z(y+1), z^3\rangle$

$\bar{r}(t) = \langle t^3, 1-3t, e^t\rangle$

$r'(t) = \langle 3t^2, -3, e^t\rangle$ $\qquad 0 \le t \le 2$

$F(r(t)) = \langle e^{2t^3}, e^t(2-3t), e^{3t}\rangle$

$\int 3t^2 e^{2t^3}$

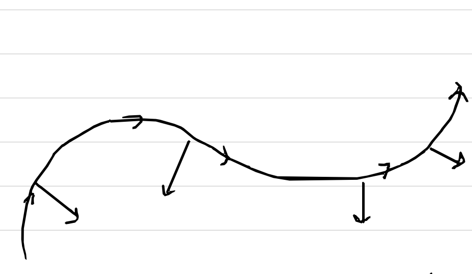
Circulation of a vector field

$$\int_C \bar{F}\cdot T\,ds$$

- A measure of how much is the vector field goes in the direction of the curve.

- Flux of a vector field
  - Direction, in / out

Towards the direction, Right is out.



$\bar{n} = \vec{T}\times\vec{K} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \bar{T}_x & \bar{T}_y & 0 \\ 0 & 0 & 1 \end{vmatrix} = T_y\hat{i} - T_x\hat{j}$

$\bar{r} = \langle \bar{x}(t), \bar{y}(t)\rangle$

$\bar{T} = \dfrac{\bar{r}'(t)}{\|\bar{r}(t)\|} = \left\langle \dfrac{x'(t)}{|r'(t)|}, \dfrac{y'(t)}{|\bar{r}'(t)|}, 0\right\rangle$

y is in front!

$\bar{n} = \left\langle \dfrac{\bar{y}'(t)}{|\bar{r}'(t)|}, \dfrac{-\bar{x}'(t)}{|\bar{r}'(t)|}\right\rangle$

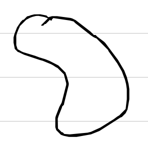
Flux $\displaystyle\int_C \bar{F}\cdot\bar{n}\,ds$

$$\int_a^b F\cdot\langle y'(t), -x'(t)\rangle\,dt$$

- 17.3 Conservative Vector Fields

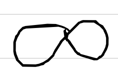
Let $\bar{r}(t)$ $\quad a \le t \le b$

Be closed $\quad \bar{r}(a) = \bar{r}(b)$

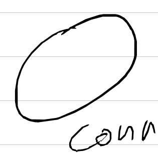


Closed simple

Not simple



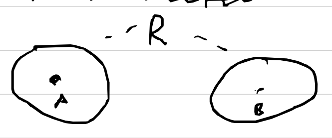
a Region in $\mathbb{R}^2$ or $\mathbb{R}^3$ is connected if any two points in region R can be connected by a continuos curve.



Connected

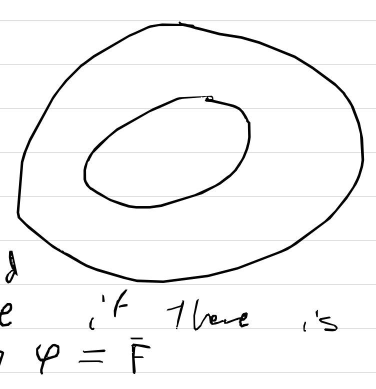
Not connected

R



Simply connected

A region in $\mathbb{R}^2$ or $\mathbb{R}^3$ is simply connected if every simple closed curve in R can be contracted to a point.



- A vector field is conservative if there is a scalar function such that $\nabla\varphi = \bar{F}$

$\bar{F} = \langle f, g, h\rangle = \nabla\varphi$

$f = \varphi_x$, $g = \varphi_y$, $h = \varphi_z$

- Theorem If F is a two dimentional vector field, then it is conservative if and only if $f_y = g_x$

For 3D

If $\bar{F}\langle f, g, h\rangle$ conservative if and only if

$$f_y = g_x \quad f_z = h_x \quad g_z = h_y$$

Prove it's conservative

just evaluate

$f_y = g_x$

Collarilly there is a φ

$\varphi = \int \bar{F}\,dx$ $\qquad \dfrac{\partial\varphi}{\partial x} = F_x$

$\varphi + C(y)$ a constant with respect to x

a function with respect to y

$\varphi_y = 2x^3 - x^2y^2 + y\sqrt{x} + C(y)$

$\dfrac{\partial p}{\partial y} = g$

$-\cancel{2xy^2} \cancel{+\sqrt{x}} + C'(y) = \cancel{-2x^2} + 4 \cancel{+\sqrt{x}}$

$C'(y) = 4$

$C(y) = 4y + D$

$\varphi(x,y) = 2x^2 - x^2y^2 + y\sqrt{x} + 4y + D$

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Journal

Lecture notes

15/5/25

For conservative fields

$$\int_C \bar{F} \cdot \bar{T}\, ds = \int_C \bar{F} \cdot d\bar{r} = \varphi(B) - \varphi(A)$$

Where poins A and B are points in R

e.g. $\int_C 2x\,dx + 2y\,dy$ where C is a smooth curve from $(0,1)$ to $(3,4)$

$\bar{F} = \langle \underset{F}{2x}, \underset{g}{2y} \rangle$ Find

line integral process

Find dr

Find $\bar{F}(dr)$

dot

integrate

$f_y = 0 \qquad g_x = 0$

Find potential function φ

$\varphi_x = 2x \qquad \varphi_y = C'(y) = 2y$

$\varphi = x^2 + C(y) \qquad \varphi = C = y^2$

$$\varphi(x,y) = (x^2 + y^2)$$

$$\int_C \bar{F} \cdot d\bar{r} = \varphi(3,4) - \varphi(0,1)$$

e.g. $\int_C \langle \cos(2y - z), -2x\sin(2y-z), x\sin(2y-z) \rangle \cdot d\bar{r}$

($\cos(2y-z) \to F$, $-2x\sin(2y-z) \to g$, $x\sin(2y-z) \to h$)

$C = \bar{r}(t) = \langle t^2, t, t \rangle, \quad 0 \le t \le \pi$

$f_y = -2\sin(2y - z) \qquad g_x = -2\sin(2y - z)$

$f_z = \sin(2y - z) \qquad h_x = \sin(2y - z)$

$g_z = 2x\cos(2y - z) \qquad h_y = 2x\cos(2y - z)$

$\bar{F} = \nabla\varphi$

$\varphi_x = \cos(2y - z)$

$\varphi = x\cos(2y - z) + C(y,z)$

Take the Derivative whit respect to y

$\varphi_y = -2x\sin(2y - z) + C(z)$

$-2x\sin(2y-z)$

$\frac{dc}{dy} = 0 \longrightarrow C(y,z) = D(z)$

$\varphi(x,y,z) = x\cos(2y-z) + D(z)$

$\varphi_z = x\sin(2y - z) + \underbrace{D'(z)}_{0} = x\sin(2y - z)$

But it dont matter

$\varphi(x,y,z) = x\cos(2y - z) + E$

Plug in to r

$r(0) = \langle 0,0,0 \rangle$

$r(\pi) = \langle \pi^2, \pi, \pi \rangle$

$$\int_C \bar{F}\, d\bar{r} = \varphi(\pi^2, \pi, \pi) - \varphi(0,0,0)$$

- Process

1 - Prove

$f_y = g_x \qquad f_z = h_x \qquad g_z = h_y$

2 - Create φ

x component of $\bar{F}$, integrate with respect to x

$\varphi = f + D(x,y)$

Derivate φ with respect to y

$\varphi_y = \ldots + D(z) = g$

if there is a difference put back?

Derivate φ with respect to z

$\varphi = \ldots + E$

$\varphi = \checkmark$

3 Find points $d\bar{r} = \langle a, b, c \rangle$ on $0 \le t \le \ldots$

evaluate,

4 then

$\varphi(B) - \varphi(A)$

Greene's Theorem 17.4

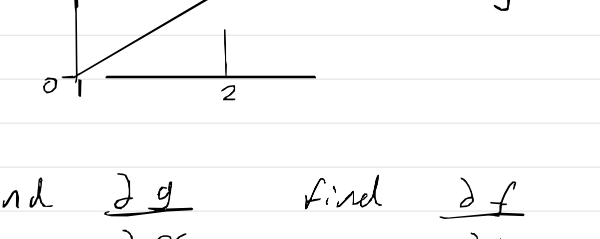
Let C be a simple and smooth piecewise curve, oriented counter clockwise. That encloses a connected and simply connected region R in $\mathbb{R}^2$. Assume we got a vector field $\bar{F}$ that are differentiable

$$\oint_C \bar{F} \cdot d\bar{r} = \oint_C f\,dx + g\,dy + h\,dz = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

e.g. use greene's theorem to evaluate

$\oint_C xy\,dx + x^2y^3\,dy$ where C is the

triangle with vectices $(0,0)$, $(1,0)$, $(1,2)$ oriented counter clockwise



① Find $\frac{\partial g}{\partial x}$ find $\frac{\partial f}{\partial y}$

$2xy^2 \qquad x$

biggest x

$$\iint 2xy^2 - x\, dA$$

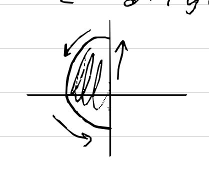
$$\int_0^1 \int_0^{2x} 2xy^2 - x\,dy\,dx$$

$$\int_0^1 \tfrac{1}{2}xy^4 - xy \Big|_0^{2x} dx$$

$$\int_0^1 8x^5 - 2x^2\,dx$$

$$\frac{4}{3}x^6 - \frac{2}{3}x^3 \Big|_0^1 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$

- e.g $\oint_C yx^2\,dx - x^2\,dy$ where C is the left semi circle of radius 5 centered at the origen



$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$

$-2x - x^2$

$$-\iint_R 2x + x^2\, dA$$

$$-\int_{\pi/2}^{3\pi/2} \int_0^5 (r\cos\theta + r^2\cos^2\theta)\, r\,dr\,d\theta$$

- Let C enclose the region R

$\oint_C x\,dy - y\,dx$

$g \to \partial y$

$f \to \partial x$

$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$

$+1 + 1 = 2 \qquad \iint_R 2\,dA$

$$A = \frac{1}{2}\oint x\,dy - y\,dx$$

Plainimeter

Divergence and Curl - 17.5

Def: $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

Let F be a vector field

The divergence of $\bar{F}$ is

$\text{div}\,\bar{F} = \nabla \cdot \bar{F}$ — At a given point, how much is the things moving away.

The curl of $\bar{F}$

$\text{curl}\,\bar{F} = \nabla \times \bar{F}$ - Out comes a vector field. with the axis of rotation of the vectors flowing round the magnitude is how strong.

e.g. $\bar{F} = \langle x^2 - y^2, xy, z \rangle$

$\text{div}\,\bar{F} = \nabla\bar{F} =$

$$\frac{\partial}{\partial x}(x^2 - y^2) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(z)$$

$= 2x + x + 1$

$F\,|r'(t)|$

$\bar{T}(r(t)) \cdot r'(t)$

$\text{curl}\,\bar{F} =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2 - y^2 & xy & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$\hat{i}\left(\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial y}(z)\right)$$

$$-\hat{j}\left(\frac{\partial}{\partial z}(x^2 - y^2) - \frac{\partial}{\partial x} z\right)$$

$$\hat{k}\left(\frac{\partial}{\partial y}(x^2 - y^2) - \frac{\partial}{\partial x}(xy)\right) = 3y\,\hat{k}$$

=

Lecture Notes 2015/25

e.g. Parametrize a cilinder with radius a and height h

$\bar{r}(u,v) = \langle a\cos u, a\sin u, v\rangle$

$0 \le u \le 2\pi \quad , \quad 0 \le v \le h$

e.g. Parametrize a sphere with radius a

$\bar{r}(u,v) = \langle a\sin(u)\cos(v), a\sin(u)\sin(v), a\cos(u)\rangle$

$0 < u \le \pi$
$0 < v \le 2\pi$

- e.g. Parametrize this paraboloid

$z = x^2 + y^2 \qquad 0 \le z \le 9$

Let $z = v$ Let $\theta = u$

$0 \le v \le 9 \qquad 0 \le u \le 2\pi$

At $z = v \Rightarrow x^2 + y^2 = z$

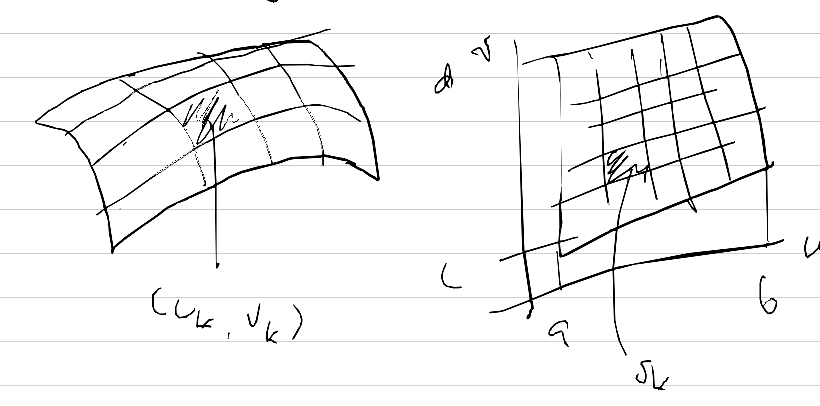
radius $\sqrt{v}$

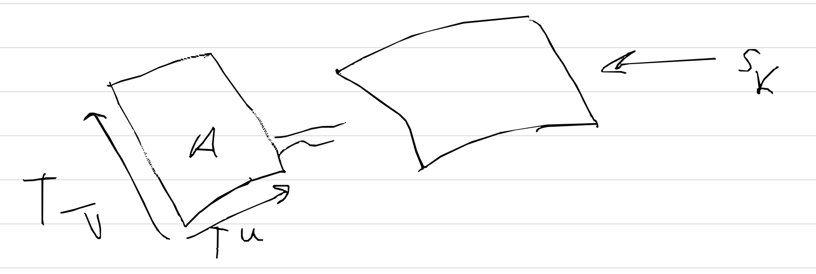
$x = \sqrt{v}\cos u$

$y = \sqrt{v}\sin u$

$\langle \sqrt{v}\cos u, \sqrt{v}\sin u, v\rangle$
$0 \le u \le 2\pi$
$0 \le v \le 9$

Surface integral





$A = \|\bar{T}_u \times \bar{T}_v\|$

- given

$\bar{r} = \langle x(u,v), y(u,v), z(u,v)\rangle$

$\bar{t}_u = \frac{\partial \bar{r}}{\partial u}$ and $\bar{t}_v = \frac{\partial r}{\partial v}$

are the tangent vectors.

The surface integral of f over S is defined as

$$\iint_S F(x,y,z)\,dS$$

Surface integral

$$= \iint_R f(x(u,v), y(u,v), z(u,v))\,\|\bar{t}_u \times \bar{t}_v\|\,dA$$

$R = \{(u,v) : a \le u \le b,\ c \le v \le d\}$

$\bar{r}(u,v) = \langle \frac{v}{\sqrt{3}}\cos(u), v^2, \frac{v}{\sqrt{3}}\sin(u)\rangle$
$R = \{(u,v) \mid 0 \le u \le 2\pi,\ 0 \le v \le \sqrt{6}\}$

$\bar{t}_u = \langle -\frac{v}{\sqrt{3}}\sin(u), 0, \frac{v}{\sqrt{3}}\cos(u)\rangle$

$\bar{t}_v = \langle \frac{1}{\sqrt{3}}\cos u, 2v, \frac{1}{\sqrt{3}}\sin u\rangle$

$$\|\bar{t}_u \times \bar{t}_v\| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} = \left| \frac{2v^2}{\sqrt{3}}\cos u\,\hat{i} + \frac{v}{3}\hat{j} - \frac{2v^2}{\sqrt{3}}\sin u\,\hat{k} \right|$$

$\sqrt{\frac{4v^4}{3} + \frac{v^2}{9}} = \sqrt{\frac{12v^4 + v^2}{9}} = \frac{1}{3}\sqrt{12v^4 + v^2}$

$\frac{v}{3}\sqrt{12v^2 + 1}$

$$\int_0^{\sqrt{6}}\int_0^{2\pi} 4v^2\,\frac{v}{3}\sqrt{12v^2+1}\,du\,dv$$

$$\frac{8\pi}{3}\int_0^{\sqrt{6}} v^2 \cdot v\sqrt{12v^2+1}$$

$w = 12v^2 + 1$
$\frac{dw}{dv} = 24v$

$$\frac{8\pi}{24 \cdot 3}\int_1^{73} \frac{w-1}{12}\sqrt{w}\,dw =$$

$$\frac{\pi}{54}\int_0^{73} w^{3/4} - w^{1/2}\,dw$$

- What if the surface is given by

$z = g(x,y)$

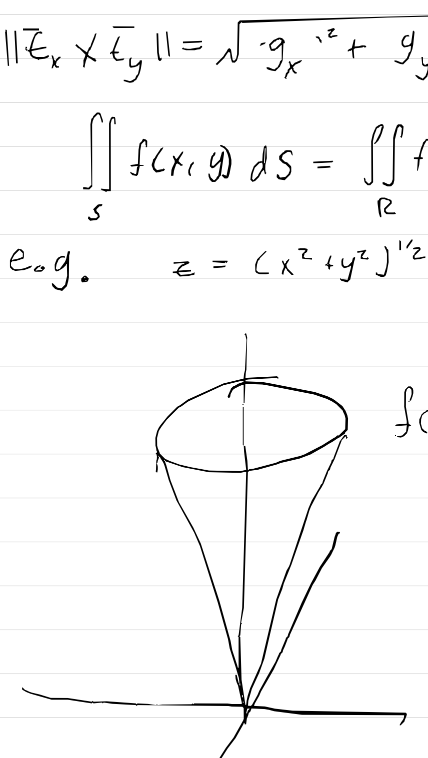
$\bar{r}(x,y) = \langle x, y, g(x,y)\rangle$

$\|\bar{t}_x \times \bar{t}_y\| = \sqrt{g_x^2 + g_y^2 + 1}$

$$\iint_S f(x,y)\,dS = \iint_R f(x, y, g(x,y))\sqrt{g_x^2 + g_y^2 + 1}\,dA$$

- e.g. $z = (x^2+y^2)^{1/2}$

$z = 0 \le z \le 4$



$f(x,y,z) = 8 - z$

$g_x = \frac{x}{\sqrt{y^2 + x^2}}$

$g_y = \frac{y}{\sqrt{y^2 + x^2}}$

$\rightarrow \sqrt{\cdots} =$

$\frac{x^2 + y^2 + x^2 + y^2}{x^2 + y^2} = 2$

$$\iint (8 - z)\,dS = \iint (8 - \sqrt{x^2+y^2})\sqrt{2}\,dA$$

$4 = \sqrt{x^2+y^2} = r = 4$

$$\sqrt{2}\int_0^{2\pi}\int_0^4 (8 - r)\,r\,dr\,d\theta$$

- Integral over vector field surface integral ← Flux

$$\iint_S \bar{F} \cdot \bar{n}\,dS = \iint_R \bar{F} \cdot (\bar{t}_u \times \bar{t}_v)\,dA$$

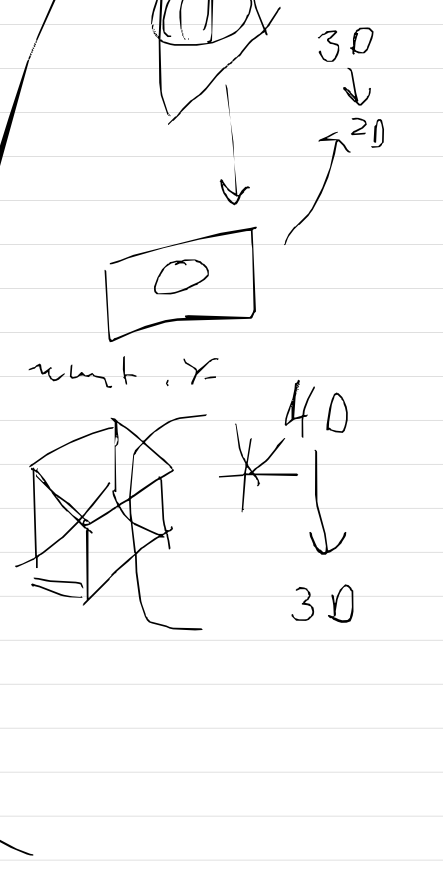
$\bar{t}_u = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial y}, \frac{\partial z}{\partial u}\rangle$

$\bar{t}_v = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial u}\rangle$

If $z = g(x,y)$

then

$$\iint_S \bar{F} \cdot \bar{n}\,dS = \iint_R (-f z_x - g z_y + h)\,dA$$



e.g. $\bar{F} = \langle x, y, z\rangle$
$\qquad\quad f \quad g \quad h$

$z^2 = x^2 + y^2,\quad 0 \le z \le 1$

normal vectors point upwards

$\bar{F} = \langle x, y, \sqrt{x^2+y^2}\rangle$

$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$

$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$

$$\iint_S \bar{F} \cdot \bar{n}\,dS = \iint \left(-\frac{x^2}{\sqrt{x^2+y^2}} - \frac{y^2}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2} \right) dA$$

$$-\iint \left(\frac{x^2 + x^2}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2} \right) dA$$

$$\int_0^{2\pi}\int_0^1 0 = 0$$

Lecture Notes

#5 Use spherical cords to evaluate

$\iiint_S x^2 \, dV$

over $x^2+y^2+z^2 = 36$

$z = -\sqrt{3x^2+3y^2}$

$\iiint (\rho\cos(\theta)\sin(\varphi))^2 \rho$

$x^2+y^2+z^2 = \rho^2 \qquad \rho = 6$

$z^2 = 3x^2+3y^2$

$(\rho\cos\varphi)^2 = 3(\rho\cos(\theta)\sin(\varphi))^2 + 3(\sin(\theta)\sin(\varphi))^2$

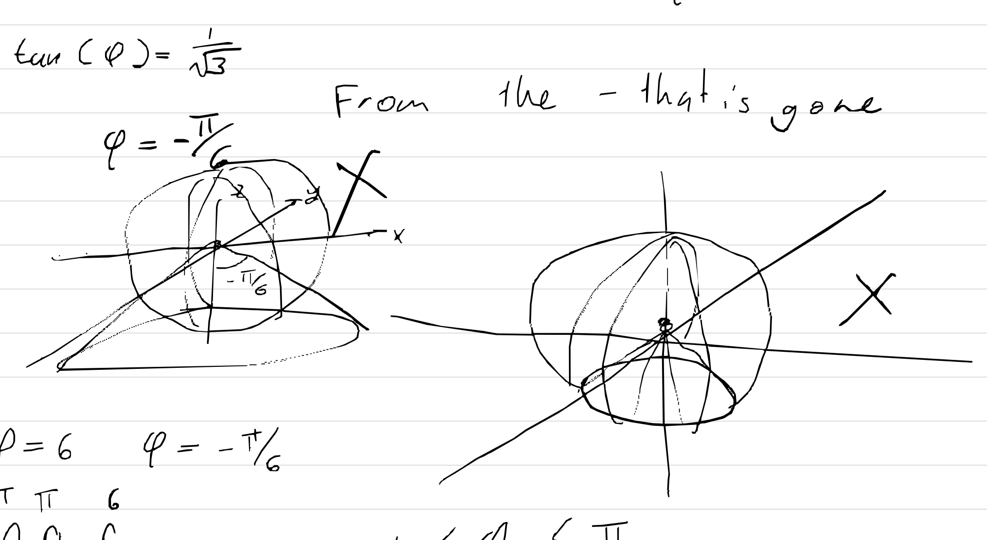
$\rho^2\cos^2\varphi = 3\rho^2\sin^2(\varphi)$

For this case is ok to remove the ρ^2

$\tan(\varphi) = \frac{1}{\sqrt{3}}$

From the − that is gone

$\varphi = -\frac{\pi}{6}$



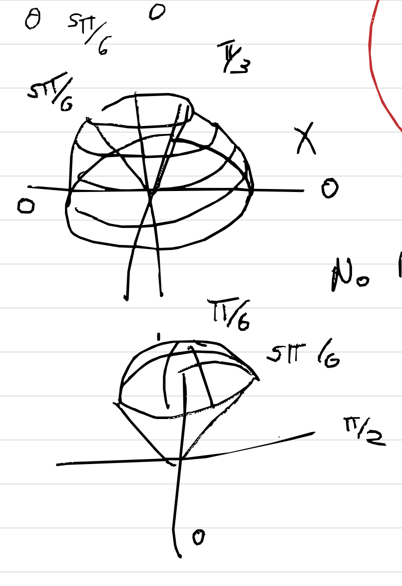
$\rho = 6 \qquad \varphi = -\pi/6$

$\int_0^{2\pi}\int_{5\pi/6}^{\pi}\int_0^{6} \dots d\dots$

$0 \le \varphi \le \pi$

$-\frac{\pi}{6} \Leftrightarrow \frac{5\pi}{6}$

Only values from $0 \le \varphi \le \pi$



No bottom?

or

we're only integrating over the cone region whith a top hat

- line integral

$\int_C x^2 dy - yz\,dz$ where C is the line segment from $(4,-1,2)$ $(1,7,-1)$

$\bar{r}(t) = \langle t^3, 1-3t, e^t\rangle$

$\bar{F} = \langle 0, (-3t+4)^2, -(8t-1)(-3t+2)\rangle$

vector field in terms of t

Not the line

$-(1-3t)(e^t)$
$(3t-1)(e^t)$

$\bar{r}'(t) = \langle 3t^2, -3, e^t\rangle$

$\bar{F}\langle 0, t^6, (3t-1)e^t\rangle$

$\int_0^1 -3t^6 + (e^t)^2(3t-1)\,dt$

10 $\int_C (y^2\cos(y+x)+y^2)dx + (2y\sin(y+x)+y^2\cos(y+x))$

$\int_C (y^2\cos(y+x)+y^2)dx + (2y\sin(y+x)+y^2\cos(y+x)+2xy-2y)\,dy$

$\nabla\vec{F} = \varphi \qquad \bar{F}$ is conservative

$\bar{F} = \langle y^2\cos(x+y)+y^2, 2y\sin(y+x)+y^2\cos(y+x)+2xy-2y\rangle$

$f_y = g_x$ Conservative

$\bar{F} = \nabla\varphi$

$\int y^2\cos(x+y) + y^2\,dx$

$y^2\sin(x+y) + xy^2 + C(y)$

$\cancel{2y\sin(x+y)} + \cancel{y^2\cos(y+x)} + \cancel{2yx} + C'(y) =$

$\cancel{2y\sin(y+x)} + \cancel{y^2\cos(y+x)} + \cancel{2xy} - 2y$

$C'(y) = -2y$

$\int -2y\,dy = -y^2$

$\varphi = y^2\sin(x+y) + y^2x - y^2$

$\int_C \bar{F}\cdot d\bar{r}, \quad \bar{r}'(t)(1-t, 2t^2-2t) \quad 0 \le t \le 2$

Plug in the endpoints

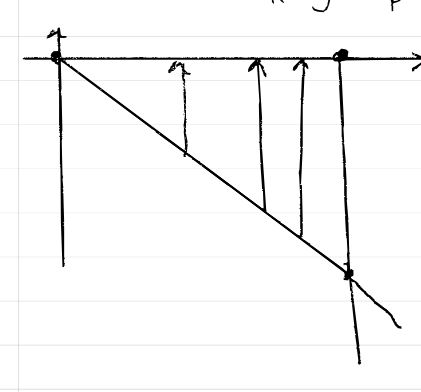
$\vec{r}(0) = \langle 1, 0\rangle \qquad \vec{r}(2) = \langle -1, 4\rangle$

$\varphi\langle -1,4\rangle - \varphi\langle 1,0\rangle = 16\sin 3 - 32$

$\langle y^2\cos(x+y)+y^2, 2y\sin(y+x)+y^2\cos(y+x)+2xy-2y\rangle$

0 0 0 0

13 $\iint_S (z+3y-x^2)\,ds$ where S the portion $z = 2-3y+x^2$ over the triangle in the xy plane $(0,0)$ $(2,0)$ $(2,-4)$



$\int_0^2\int_{-2x}^0 dy\,dx$

$y = -2x$

$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,g(x,y))\sqrt{g_x^2+g_y^2+1}\,dA$

$g(x,y) = 2-3y+x^2$
$f(x,y,z) = z+3y-x^2$

$f(x,y,g(x,y)) = 2 \cancel{-3y} \cancel{+x^2} \cancel{+3y} \cancel{-x^2}$

$f(x,y,g(x,y)) = 2$

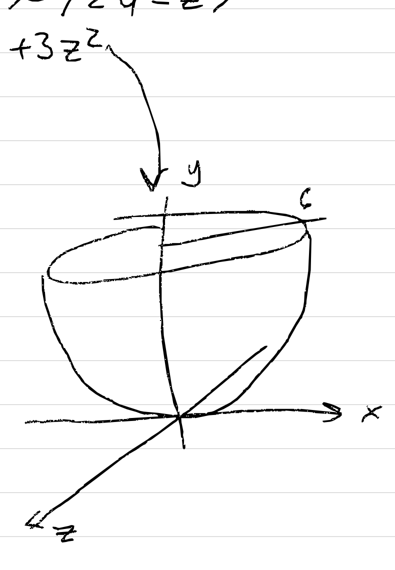
$g_x = 2x \qquad g_y = -3$

$\sqrt{4x^2+9+1} = \int_0^2\int_{-2x}^0 2\sqrt{4x^2+9+1}\,dy\,dx$

- 14 Flux of the vector field

$\bar{F} = \langle -x, 2y-z\rangle$
$y = 3x^2+3z^2$
$y = 6$

$\boxed{\iint \bar{F}\cdot\bar{n}\,ds \text{ Flux}}$



Parametrize

where

$\langle a\cos u, v^2, a\sin u\rangle$

$\langle \frac{v}{\sqrt{3}}\cos u, v^2, \frac{v}{\sqrt{3}}\sin u\rangle$

$v^2 = 3a^2$

$0 \le u \le 2\pi$
$0 \le v \le \sqrt{6}$

$\frac{v^2}{3} = a^2 = \frac{v}{\sqrt{3}} = a$

$\vec{r}_u = \langle -\frac{v}{\sqrt{3}}\sin u, 0, \frac{v}{\sqrt{3}}\cos u\rangle$

$\bar{r}_v = \langle \frac{\cos u}{\sqrt{3}}, 2v, \frac{\sin u}{\sqrt{3}}\rangle$

$\bar{r}_u \times \bar{r}_v = \langle -\frac{2v^2\cos(u)}{\sqrt{3}}, \frac{1}{3}v, -\frac{2v^2\sin u}{\sqrt{3}}\rangle$

Verify that the vector points towards the right direction positive y dir

at $(0,1) \Rightarrow \langle -2/\sqrt{3}, 1/3, 0\rangle$

yes Bc $1/3 \ge 0$

Orientation is important

$\iint_C \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}(\bar{r}(u,v))\cdot(\vec{r}_u\times\bar{r}_v)\,dA$

$\bar{F}(\bar{r}'(t)) = \langle -\frac{v}{\sqrt{3}}\cos u, 2v^2, -\frac{v}{\sqrt{3}}\sin u\rangle$

$\langle -\frac{2v^2\cos(u)}{\sqrt{3}}, \frac{1}{3}v, -\frac{2v^2\sin u}{\sqrt{3}}\rangle$

$\iint \langle -\frac{v}{\sqrt{3}}\cos(u), 2v^2, -\frac{v}{\sqrt{3}}\sin u\rangle \cdot$

$\langle -\frac{2v^2\cos(u)}{\sqrt{3}}, \frac{1}{3}v, -\frac{2v^2\sin u}{\sqrt{3}}\rangle$

$\int_0^{\sqrt{6}}\int_0^{2\pi} \frac{2v^3}{3}\cos^2(u) + \frac{2}{3}v^3 + \frac{2v^3}{3}\sin^2 u\,du\,dv$

$\int_0^{\sqrt{6}}\int_0^{2\pi} \frac{2v^3}{3} + \frac{2}{3}v^3 = \frac{4}{3}\int_0^{\sqrt{6}}\int_0^{2\pi} v^3\,du\,dv$

1 - Parametrize
2 - Partial derivatives
3 - Cross product
3 - Verify
3 - $\bar{F}(t(t)) \cdot t_u \times t_v$
4 - Integrate $\iint$

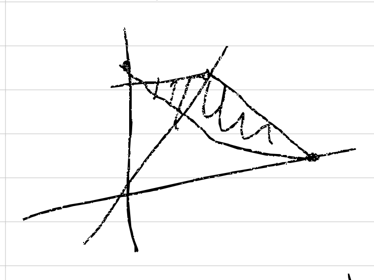
- 16

$\int_C \bar{F}\,d\bar{r} \quad . \quad \bar{F} = z^2\hat{i} + y^2\hat{j} + x\hat{k}$

$\iint(\nabla\times F)\cdot\bar{n}\,ds$ triagle with veurtices $(1,0,0)$ $(0,1,0)$, $(0,0,1)$

P A B

Plain



$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$\langle -1,1,0\rangle$
$\langle -1,0,1\rangle \qquad \hat{i}+\hat{j}+\hat{k}$ -

Plain $\int \bar{F}\cdot\bar{n} = \iint F(\bar{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,dA$

$(x-1) + y + z = 0$
$z = 1-x-y$

Curl $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix}$

lif your parametization is $(u, v, g(x,y))$

$\downarrow$

$r_u \times r_v$

0
$+\hat{j}$

curl $\bar{F} = \langle 0, 2z-1, 0\rangle$

$\downarrow$

$\langle g_x, g_y, 1\rangle$

$z = g(x,y)$

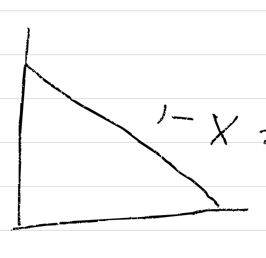
$\bar{r}(u,v) = \langle u, v, g(u,v)\rangle$

$\iint (\nabla\cdot\bar{F})\cdot\bar{n}\,ds = \iint_R \langle 0, 2-2x-2y, 0\rangle\cdot\langle 1,1,1\rangle\,ds$

$\nabla\cdot F = \langle 0, 2z-1, 0\rangle$

$z = \underbrace{1-x-y}_{g(x,y)}$

$\int_0^1\int_0^{1-x} 2-2x-2y\,dy\,dx$



$1-x = y$

- 17 $\bar{F} = \langle yx^2, xy^2-3z^4, x^3+y^2\rangle$

$\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_D \nabla\cdot\bar{F}\,dV$

$2xy + 2xy + 0 = 4xy = \nabla\cdot\bar{F} \qquad z \le 0$
$y \le 0$

$\iiint 4xy$

$\iiint 4(\rho\sin(\varphi)\cos(\theta))(\rho\sin(\varphi)\sin(\theta))\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

$\int_{\pi/2}^{\pi}\int_{\pi/2}^{3\pi/2}\int_0^4 4\rho^4\sin^2\varphi\cos^2\theta\sin\theta\,dr\,d\varphi\,d\theta$

1 $\nabla\cdot F$
2 How to integrate in 3D