Practice Problems for Exam 2

1. Find all first- and second-order derivatives of the following functions.
   (a) $h(x, y) = 5x^3y + y^4$
   (b) $I(x, y) = \sin\left(\frac{x}{y}\right)$
   (c) $F(x, y, z) = x\cos\left(y + z^2\right)$
2. Determine $\frac{\mathrm{d}y}{\mathrm{d}x}$, given y as a function of $u(x)$ and $v(x)$. (And $w(x)$ if w is present)
   (a) $y = u^3v - \sin u$, $u = 2x$, $v = x^2$
   (b) $y = \sin^{-1}\left(\frac{u}{v}\right)$, $u = 2x$, $v = x^3$
   (c) $y = uve^w$, $u = 5x - 2$, $v = \sin x$, $w = \ln x$
3. Use the chain rule to determine $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$
   (a) $z = \cos\left(u + v^2\right)$, $u = y - \frac{x^2}{2}$, $v = 3x - 2y$
   (b) $z = v^3 - 2u^2v$, $u = \cos y$, $v = \sin x$
4. Find the derivative of f at the given point in the given direction.
   (a) $f(x, y) = 3x^2 - 2xy + y^2$ at $P(0, 1)$ in the direction of $\vec{v} = \langle 1, 1\rangle$
   (b) $f(x, y, z) = e^{-\left(x^2+y^2+z^2\right)}$ at $P(2, 1, 1)$ in the direction of $\vec{v} = \langle -6, 2, 3\rangle$
5. Find the direction and value of the greatest rate of increase for the following functions at the given points.
   (a) $f(x, y) = x^2e^{2xy}$ at $(1, 0)$
   (b) $f(x, y, z) = \frac{x - y}{z + 2}$ at $(4, 2, 0)$
6. Determine the tangent plane of $z^2 - 3x^2 - y^2 = 15$ at $(2, -3, 6)$
7. Use the second derivative test to classify the critical points of the following functions.
   (a) $f(x, y) = x^3 + y^3 - 3x^2 - 2y^2$
   (b) $f(x, y) = ye^{x^2-y^2}$
8. Find the extreme values of the function subject to the given constraint.
   (a) $f(x, y) = x^2 + 3xy + y^2$ subject to $y - x = 2$.
   (b) $f(x, y) = 2xy$ subject to $\frac{x^2}{9} + \frac{y^2}{4} = 1$
   (c) $f(x, y, z) = x + 2y + 3z$ subject to $x^2 + y^2 + z^2 = 1$
9. Evaluate the following integrals
   (a) $\int_{-1}^{0}\int_{0}^{3}\left(xy^4 - x^4y\right)dx\,dy$

(b) $\displaystyle\int_{-3/2}^{0}\int_{0}^{\pi}\left(x^2y-\cos x\right)dx\,dy$

(c) $\displaystyle\int_{0}^{2}\int_{x/2}^{3-x} xy\,dy\,dx$

(d) Same as (c), but reverse the order of integration.

(e) $\displaystyle\int_{0}^{1}\int_{y^2}^{y} dx\,dy$

(f) Same as (e), but reverse the order of integration.

10. Evaluate the following integrals. You may want polar coordinates for some of these.

(a) $\displaystyle\iint_R y^2\,\mathrm{d}A$ where R is the region bounded by $x=1$, $y=2x+2$, and $y=-x-1$

(b) $\displaystyle\iint_R x^2y\,\mathrm{d}A$ where R is the region in quadrants 1 and 4 bounded by semicircle of radius 4 centered at (0,0).

(c) $\displaystyle\iint_R (x+y)\,\mathrm{d}A$ where R is the region bounded by $y=1/x$ and $y=5/2-x$

11. Use a double integral in polar coordinates to find the area of the region inside $r=1-\sin\theta$.

12. Evaluate the following integral by changing to polar coordinates.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\cos\left(x^2+y^2\right)$$

13. Evaluate the following integrals

(a) $\displaystyle\iiint_S xy^3z\,dV$ where $S=[-1,3]\times[0,1]\times[1,3]$

(b) $\displaystyle\int_{0}^{3}\int_{1}^{2}\int_{0}^{x+3y}(x+y)\,dz\,dy\,dx$

Answer Key

1. (a) First order:
    - $h_x(x,y) = 15x^2 y$
    - $h_y(x,y) = 4y^3 + 5x^3$

    Second order:
    - $h_{xx}(x,y) = 30xy$
    - $h_{xy}(x,y) = h_{yx}(x,y) = 15x^2$
    - $h_{yy}(x,y) = 12y^2$

    (b) First order:
    - $I_x(x,y) = \dfrac{\cos\left(\frac{x}{y}\right)}{y}$
    - $I_y(x,y) = -\left(\dfrac{x\ \cos\left(\frac{x}{y}\right)}{y^2}\right)$

    Second order:
    - $I_{xx}(x,y) = -\left(\dfrac{x\ \cos\left(\frac{x}{y}\right)}{y^2}\right)$
    - $I_{xy}(x,y) = I_{yx}(x,y) = \dfrac{x\ \sin\left(\frac{x}{y}\right)}{y^3} - \dfrac{\cos\left(\frac{x}{y}\right)}{y^2}$
    - $I_{yy}(x,y) = \dfrac{2\,x\ \cos\left(\frac{x}{y}\right)}{y^3} - \dfrac{x^2\ \sin\left(\frac{x}{y}\right)}{y^4}$

    (c) First order:
    - $F_x = \cos\left(z^2 + y\right)$
    - $F_y = -\left(x\ \sin\left(z^2 + y\right)\right)$
    - $F_z = -\left(2\,x\,z\ \sin\left(z^2 + y\right)\right)$

    Second order:
    - $F_{xx} = 0$
    - $F_{yy} = -\left(x\ \cos\left(z^2 + y\right)\right)$
    - $F_{zz} = -\left(2\,x\ \sin\left(z^2 + y\right)\right) - 4\,x\,z^2\ \cos\left(z^2 + y\right)$
    - $F_{xy} = F_{yx} = -\sin\left(z^2 + y\right)$
    - $F_{xz} = F_{zx} = -\left(2\,z\ \sin\left(z^2 + y\right)\right)$
    - $F_{yz} = F_{zy} = -\left(2\,x\,z\ \cos\left(z^2 + y\right)\right)$

2. (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \left(3u^2 v - cos(u)\right)(2) + \left(u^3\right)(2x)$

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \left(\dfrac{1}{v\sqrt{1-\frac{u^2}{v^2}}}\right)(2) - \left(\dfrac{u}{v^2\sqrt{1-\frac{u^2}{v^2}}}\right)(3x^2)$

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = ve^{w}(5) + ue^{w}\cos x + uve^{w}\left(\dfrac{1}{x}\right)$

3. (a) $\dfrac{\partial z}{\partial x} = -\sin(u+v^2)(x) - 2v\sin(u+v^2)(3)$

   $\dfrac{\partial z}{\partial y} = -\sin(u+v^2) - 2v\sin(u+v^2)(-2)$

   (b) $\dfrac{\partial z}{\partial x} = \left(3v^2 - 2u^2\right)\cos x$

   $\dfrac{\partial z}{\partial y} = -4uv(-\cos y)$

4. Find the derivative of f at the given point in the given direction.

   (a) 0

   (b) $2e^{-6}$

5. Find the direction and value of the greatest rate of increase for the following functions at the given points.

   (a) $\langle 2, 2\rangle$

   (b) $\left\langle \dfrac{1}{2}, -\dfrac{1}{2}, \dfrac{1}{2}\right\rangle$

6. $-12(x-2) + 6(y+3) + 12(z-6) = 0$

7. (a) Local maximum: $(x, y) = (0, 0)$

   Local minimum: $(x, y) = \left(2, \dfrac{4}{3}\right)$

   (b) Local maximum: None

   Local minimum: None

   Saddle points at $(x, y) = \left(0, \pm\dfrac{1}{\sqrt{2}}\right)$

8. (a) Maximum: None

   Minimum: $(x, y) = (-1, 1)$

   (b) Maximum: $\left(-\dfrac{3}{\sqrt{2}}, -\sqrt{2}\right), \left(\dfrac{3}{\sqrt{2}}, \sqrt{2}\right)$

   Minimum: $\left(-\dfrac{3}{\sqrt{2}}, \sqrt{2}\right), \left(\dfrac{3}{\sqrt{2}}, -\sqrt{2}\right)$

   (c) Maximum: $\left(\dfrac{1}{\sqrt{14}}, \sqrt{\dfrac{2}{7}}, \dfrac{3}{\sqrt{14}}\right)$

   Minimum: $\left(-\dfrac{1}{\sqrt{14}}, -\sqrt{\dfrac{2}{7}}, -\dfrac{3}{\sqrt{14}}\right)$

9. Evaluate the following integrals

(a) $\frac{126}{5}$

(b) $-\frac{3\pi^3}{8}$

(c) $\frac{5}{2}$

(d) $\frac{5}{2}$

(e) $\frac{1}{6}$

(f) $\frac{1}{6}$

10. (a) 12

    (b) 0

    (c) $\frac{9}{8}$

11. $\frac{3\pi}{2}$

12. $\pi \sin(4)$

13. (a) 4

    (b) 57